Sehee Lim

1.10

$(2n+1)+(2n+3)+\cdots+(4n-1)=3n^2$ for $\forall n>0$.

1) $n=1$

$3=3$; Clear

2) Assume that $(2n+1)+(2n+3)+\cdots+(4n-1)=3n^2$ is true when $n=k$.

$$(2k+1)+(2k+3)+\cdots+(4k-1)=3k^2$$

$$(2k+1)+(2k+3)+\cdots+(4k-1)+(4k+1)+(4k+3)=3k^2+(4k+1)+(4k+3)$$

$$(2k+3)+\cdots+(4k-1)+(4k+1)+(4k+3)=3k^2+(4k+1)+(4k+3)-(2k+1)$$

$$(2(k+1)+1)+\cdots+(4(k)-1)+(4(k+1)-1)=3k^2+6k+3=3(k+1)^2$$

Then, it is also true when $n=k+1$

By mathematical induction, $(2n+1)+(2n+3)+\cdots+(4n-1)=3n^2$ is true for all positive integers n.

1.12

(a) 1) $n=1$

$(a+b)^1=\binom{1}{0}a^1+\binom{1}{1}b^1=a+b$

~~2) $n=2$~~

~~$(a+b)^2=\binom{2}{0}a^2+\binom{2}{1}ab+\binom{2}{2}b^2=a^2+2ab+b^2$~~

~~3) $n=3$~~

~~$(a+b)^3=\binom{3}{0}a^3+\binom{3}{1}a^2b+\binom{3}{2}ab^2+\binom{3}{3}b^3=a^3+3a^2b+3ab^2+b^3$~~

(b) $\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$ for $k=1,2,\cdots,n$

$$\frac{n!}{k!(n-k)!}+\frac{n!}{(k-1)!(n-k+1)!}=\frac{(n+1)!}{k!(n-k+1)!}$$

$$\frac{\cancel{n!}}{\cancel{k!}(n-k)!}\times\frac{\cancel{k!}(n-k+1)!}{\cancel{n!}}+\frac{\cancel{n!}}{(k-1)!\cancel{(n-k+1)!}}\times\frac{k!\cancel{(n-k+1)!}}{\cancel{n!}}=\frac{(n+1)!}{\cancel{k!}\cancel{(n-k+1)!}}\times\frac{\cancel{k!}\cancel{(n-k+1)!}}{n!}$$

(Multiple both sides by $\frac{k!(n-k+1)!}{n!}$)

$(n-k+1)+(k)=n+1$; Clear

(c) $(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \cdots + \binom{n}{n} b^n$

1) $n=1$

$(a+b)^1 = \binom{1}{0} a^1 + \binom{1}{1} b^1 = a+b$; Clear

2) Assume that $(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \cdots + \binom{n}{n} b^n$ is true.

$(a+b)^{n+1}$

$= (a+b)(a+b)^n$

$= (a+b)\left(\binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \cdots + \binom{n}{n} b^n\right)$

$= \binom{n}{0} a^{n+1} + \binom{n}{1} a^n b + \cdots + \binom{n}{n} ab^n + \binom{n}{0} a^n b + \binom{n}{1} a^{n-1}b^2 + \cdots + \binom{n}{n} b^{n+1}$

$= \binom{n}{0} a^{n+1} + \left(\binom{n}{1} + \binom{n}{0}\right) a^n b + \cdots + \left(\binom{n}{n} + \binom{n}{n-1}\right) ab^n + \binom{n}{n} b^{n+1}$

$= \binom{n}{0} a^{n+1} + \binom{n+1}{1} a^n b + \cdots + \binom{n+1}{n} ab^n + \binom{n}{n} b^{n+1}$

$= \binom{n+1}{0} a^{n+1} + \binom{n+1}{1} a^n b + \cdots + \binom{n+1}{n} ab^n + \binom{n+1}{n+1} b^{n+1}$

H is also true for $n+1$.

By mathematical induction, $(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \cdots + \binom{n}{n} b^n$ is true.

#2.1

1) $\sqrt{3}$ is not a rational number

The only possible rational solutions of $x^2-3=0$ are $\pm 1, \pm 3$.
None of these numbers are solutions.

2) $\sqrt{5}$ is not a rational number.

The only possible rational solutions of $x^2-5=0$ are $\pm 1, \pm 5$.
None of these numbers are solutions.

3) $\sqrt{7}$ is not a rational number.
The only possible rational solutions of $x^2-7=0$ are $\pm 1, \pm 7$.
None of these numbers are solutions.

4) $\sqrt{24}$ is not a rational number.

The only possible rational solutions of $x^2-24=0$ are $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$.
None of these numbers are solutions.

5) $\sqrt{31}$ is not a rational number.

The only possible rational solutions of $x^2-31=0$ are $\pm1, \pm31$.
None of these numbers are solutions.

#2.2

1) $\sqrt[3]{2}$ is not a rational number.

The only possible rational solutions of $x^3-2=0$ are $\pm1, \pm2$.
None of these numbers are solutions.

2) $\sqrt[7]{5}$ is not a rational number.

The only possible rational solutions of $x^7-5=0$ are $\pm1, \pm5$.
None of these numbers are solutions.

3) $\sqrt[4]{13}$ is not a rational number.

The only possible rational solutions of $x^4-13=0$ are $\pm1, \pm13$.
None of these numbers are solutions.

#2.7

(a) $\sqrt{4+2\sqrt{3}}-\sqrt{3}$ is actually a rational number.

$\sqrt{4+2\sqrt{3}}-\sqrt{3} = \sqrt{(1+\sqrt{3})^2}-\sqrt{3} = 1+\sqrt{3}-\sqrt{3} = 1$

1 is a rational number.

(b) $\sqrt{6+4\sqrt{2}}-\sqrt{2}$ is actually a rational number.

$\sqrt{6+4\sqrt{2}}-\sqrt{2} = \sqrt{(2+\sqrt{2})^2}-\sqrt{2} = 2+\sqrt{2}-\sqrt{2} = 2$

2 is a rational number.

3.6

(a) ~~$|a+b+c| \le |a|+|b|+|c|$~~

Triangle Inequality $|a+b| \le |a|+|b|$

$$|(a+b)+c| \underset{1}{\le} |a+b|+|c| \underset{2}{\le} |a|+|b|+|c|$$

(b) $|a_1+a_2+\cdots+a_n| \le |a_1|+|a_2|+\cdots+|a_n|$

~~1) $n=1$~~

~~$|a_1| \le |a_1|$; Clear~~

2) Assume that $|a_1+a_2+\cdots+a_k| \le |a_1|+|a_2|+\cdots+|a_k|$ is true.

$|a_1+a_2+\cdots+a_k+a_{k+1}| \le |a_1+\cdots+a_k|+|a_{k+1}| \le |a_1|+|a_2|+\cdots+|a_k|+|a_{k+1}|$

It is also true for $k+1$.

By mathematical induction, $|a_1+a_2+\cdots+a_n| \le |a_1|+|a_2|+\cdots+|a_n|$ is true.

4.11

Suppose there are n finite rationals between a and b.

$(a < x_1 < x_2 \cdots < x_n < b)$

By denseness of $\mathbb{Q}$, there is a rational r such that $x_n < r < b$.

Since there are $(n+1)$ finite rationals between a and b,

it is contradiction.

Therefore, there are infinitely many rationals between a and b.

4.14

(a) ~~$a \le \sup(A)$ for $\forall a \in A$~~

~~$b \le \sup(B)$ for $\forall b \in B$~~

$a+b \le \sup(A+B)$ for $\forall a+b \in A+B$

1. $\sup(A)+\sup(B) \le \sup(A+B)$

I'll use "least upper bound $\le$ upper bound".

$a \le \sup(A+B)-b$

$\Rightarrow a \le \underbrace{\sup(A) \le \sup(A+B)-b}$

$b \le \sup(A+B) - \sup(A)$

$\Rightarrow \quad b \le \sup(B) \le \sup(A+B) - \sup(A)$

$\sup(A) + \sup(B) \le \sup(A+B)$

2) $\sup(A+B) \le \sup(A) + \sup(B)$

$a \le \sup(A)$

$b \le \sup(B)$

$\Rightarrow \quad a+b \le \sup(A) + \sup(B)$

Since "least upper bound $\le$ upper bound",

$a+b \le \sup(A+B) \le \sup(A) + \sup(B)$

Since $\sup(A) + \sup(B) \le \sup(A+B)$

$\sup(A+B) \le \sup(A) + \sup(B)$, $\quad \sup(A) + \sup(B) = \sup(A+B)$.

(b) $\inf(A+B) = \inf(A) + \inf(B)$

$\inf(A) \le a$ for $\forall a \in A$

$\inf(B) \le b$ for $\forall b \in B$

$\inf(A+B) \le a+b$ for $\forall a+b \in A+B$

1. $\inf(A) + \inf(B) \ge \inf(A+B)$

$\inf(A+B) - b \le a$

Since "greatest lower bound $\ge$ lower bound", $\inf(A+B) - b \le \inf(A)$

$\inf(A+B) - \inf(A) \le b$

$\inf(A+B) - \inf(A) \le \inf(B)$

$\Rightarrow \inf(A+B) \le \inf(A) + \inf(B)$

2. $\inf(A+B) \le \inf(A) + \inf(B)$

$\inf(A) \le a$

$\inf(B) \le b$

$\Rightarrow \inf(A) + \inf(B) \le a+b$

Since "greatest lower bound $\ge$ lower bound", $\inf(A) + \inf(B) \le \inf(A+B)$

Since $\inf(A+B) \le \inf(A) + \inf(B)$,

$\inf(A+B) \ge \inf(A) + \inf(B)$. $\quad \inf(A) + \inf(B) = \inf(A+B)$

PANTONE

7.5

(a) $\lim_{n\to\infty}(\sqrt{n^2+1}-n)$

$$= \lim_{n\to\infty}\frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n}$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{n^2+1}+n} = 0 \qquad \therefore 0$$

(b) $\lim_{n\to\infty}(\sqrt{n^2+n}-n)$

$$= \lim_{n\to\infty}\frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n}$$

$$= \lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}+n} = \frac{1}{2} \qquad \therefore \frac{1}{2}$$

(c) $\lim_{n\to\infty}(\sqrt{4n^2+n}-2n)$

$$= \lim_{n\to\infty}\frac{(\sqrt{4n^2+n}-2n)(\sqrt{4n^2+n}+2n)}{\sqrt{4n^2+n}+2n}$$

$$= \lim_{n\to\infty}\frac{n}{\sqrt{4n^2+n}+2n} = \frac{1}{2+2} \qquad \therefore \frac{1}{4}$$