Ross 9.9

Given that $(s_n), (t_n)$ are sequences such that in the tail, $s_n \leq t_n$, we'd like to prove the following:

a. $\lim s_n = +\infty \implies \lim t_n = +\infty$. Suppose for contradiction $\lim t_n$ is finite. Fix some $\epsilon > 0$. Then $\exists N$ s.t. $\forall n > N$:

$$t_n < t + \epsilon$$

Since s_n does not converge, we know that for every $\epsilon' > 0$, there is an N' such that $\forall n' > N'$:

$$s_{n'} > \epsilon$$

Then let $\epsilon' = t + \epsilon$, ad let $M = \max(N, N')$. Then we get that:

$$s_n > t + \epsilon > t_n$$

which is a contradiction. We could also (for contradiction) have that $\lim t_n = -\infty$, but in that case we have that $t_n < \epsilon$, and we let $\epsilon' = \epsilon$ to again see that $s_n > t_n$, contradicting our earlier assumption.

b. $\lim t_n = -\infty \implies \lim s_n = -\infty$. We can do this proof in essentially the same way. Suppose for contradiction $\lim s_n$ is finite. Fix some $\epsilon > 0$. Then $\exists N$ s.t. $\forall n > N$:

$$s_n > s - \epsilon$$

Since t_n diverges to $-\infty$, we know that for every $\epsilon' > 0$, there is an N' such that $\forall n' > N'$:

$$t_{n'} < \epsilon$$

Then let $\epsilon' = s - \epsilon$, ad let $M = \max(N, N')$. Then we get that:

$$s_n > s - \epsilon > t_n$$

which is a contradiction. We could also (for contradiction) have that $\lim s_n = +\infty$, but in that case we have that $s_n > \epsilon$, and we let $\epsilon' = \epsilon$ to again see that $s_n > t_n$, contradicting our earlier assumption.

c. $\lim s_n \leq \lim t_n$. Suppose for contradiction $\lim s_n > \lim t_n$ (call these limits s, t respectively), and fix some $\epsilon > 0$. There is N such that $\forall n > N$:

$$s_n > s - \epsilon$$

$$s_n > t - \epsilon$$

now we let $\epsilon = t - t_n$

$$s_n > t - (t - t_n) = t_n$$

contradicting our original assumption.

Ross 9.15

We'd like to show that $\lim_{n\to\infty} \frac{a^n}{n!} = 0$. We can split this up into 3 cases. The case where $a = 0, a > 0, a < 0$. If $a = 0$, the result is trivial, as $\frac{a^0}{n!} = 0$ for all n. Now let's consider the case where $a > 0$. Then $2a > a$, and for any $n > 2a$, we have that $\frac{a}{n} \leq \frac{a}{2a}$. Since we can start the limit from any arbitrary point (we only care about the tail), we can see that if n starts at something larger than $2a$:

$$\lim_{n\to\infty} \frac{a^n}{n!} \leq \lim_{n\to\infty} \frac{a^n}{(2a)^n} = \lim_{n\to\infty} \frac{1}{2^n} = 0$$

We can use a very similar argument to see that if $a < 0$, then $-2a > a$, and for any $n > -2a$, we have that $\frac{a}{n} \leq \frac{a}{-2a}$, which means that (again starting at some $n > -2a$):

$$\lim_{n\to\infty} \frac{a^n}{n!} \leq \lim_{n\to\infty} \frac{a^n}{(-2a)^n} = \lim_{n\to\infty} \frac{(-1)^n}{2^n} = \lim_{n\to\infty} (-1)^n \lim_{n\to\infty} \frac{1}{2^n} = 0$$

In all 3 cases, we can see that the limit is 0.

Ross 10.7

Given a subset S such that $\sup S \notin S$, we'd like to show that there's a sequence (s_n) in S such that $\lim s_n = \sup S$. Intuitively, we'd like to set the $s_n = \sup S - \frac{1}{n}$ or something, but that's not always possible because it's not guaranteed that $\sup S - \frac{1}{n}$ is going to be an element of S. Instead we let s_n be something chosen from the interval $(\sup S - \frac{1}{n}, \sup S)$, which we can do using the axiom of choice. We can show that the limit is $\sup S$ in the way that we normally would.

$$\sup S - \frac{1}{n} - \sup S < s_n - \sup S < \sup S - \sup S$$

We can do that by construction of s_n:

$$-\frac{1}{n} < s_n - \sup S < 0$$

We can rearrange this to see that $s_n - \sup S < \frac{1}{n}$, which means that we for arbitrary $\epsilon > 0$ we let N_0 be the ceiling of $\frac{1}{\epsilon}$, and then we have that for any $\epsilon > 0$, there is an N_0 such that for all $n > N_0$: $s_n - \sup S \leq \epsilon$

Ross 10.8

We'd like to show that given an increasing sequence of positive numbers (s_n), the arithmetic mean σ_n of the first n values of s is an increasing sequence. To show that a sequence is increasing, we basically have to show that $\sigma_{n+1} \geq \sigma_n$. Suppose for contradiction $\sigma_{n+1} < \sigma_n$. Then:

$$\frac{s_1 + ... + s_{n+1}}{n+1} < \frac{s_1 + ... + s_n}{n}$$
$$\implies n(s_1 + ... + s_n) + ns_{n+1} < n(s_1 + ... + s_n) + (s_1 + ... + s_n)$$
$$\implies ns_{n+1} < s_1 + ... + s_n$$

However, since we said that (s_n) was an increasing sequence, we know that $s_{n+1} \geq s_i$ for all $i \in 1, 2, ..., n$, which means that ns_{n+1} should be greater than or equal to $s_1 + ... + s_n$. This contradiction implies that $\sigma_{n+1} \geq \sigma_n$, i.e. (σ_n) is an increasing sequence.

Ross 10.9

a.

$$s_2 = \frac{2}{3}1^2 = \frac{2}{3}$$
$$s_3 = \frac{3}{4}(\frac{2}{3})^2 = \frac{1}{3}$$
$$s_4 = \frac{4}{5}(\frac{1}{3})^2 = \frac{4}{45}$$

b. Firstly, we can see that (s_n) is bounded below by 0, because $\frac{n}{n+1} > 0$ for $n > 0$, and of course $s_n^2 \geq 0$ for all s_n. Secondly, note that $s_2 \leq 1$, which means that for all $n \geq 2$, we have that $s_n \leq 1$. We can prove this by induction. Suppose $s_n \leq 1$, then we know that $\frac{n}{n+1} \leq 1$, and $s_n^2 \leq 1$, which means that their product is also less than 1, therefore $s_{n+1} \leq 1$. So we can see the (s_n) is bounded above by 1. Secondly, we claim that it's a decreasing sequence.

$$s_{n+1} = \frac{n}{n+1}s_n^2 \leq s_n^2 \leq s_n$$

where the last equality follows from the fact that since $s_n \leq 1$, $s_n^2 \leq s_n$.

c. Now we'd like to show that the limit is 0. We know that the sequence is bounded below by 0, so we want to find a sequence (a_n) such that $0 \leq s_n \leq a_n$ such that the limit of (a_n) is 0. For $n > 2$, we can guess that $s_n \leq \frac{1}{n}$. We can look at s_3 as a base case. Then suppose $s_n \leq \frac{1}{n}$:

$$s_{n+1} = \frac{n}{n+1}s_n^2 \leq s_n^2 \leq \frac{1}{n^2} \leq \frac{1}{n+1}$$

The last inequality just comes from the fact that certainly for $n \geq 3$, we have $n^2 > n+1$. Since we know that $\lim \frac{1}{n} = 0$, we can conclude that $\lim s_n = 0$.

Ross 10.10

a.

$$s_2 = \frac{2}{3}$$
$$s_3 = \frac{5}{9}$$
$$s_4 = \frac{14}{27}$$

b. $s_1 > \frac{1}{2}$ by definition. We proceed with induction, so suppose that $s_n > \frac{1}{2}$

$$s_{n+1} = \frac{1}{3}(s_n + 1) > \frac{1}{3}(\frac{1}{2} + 1) = \frac{1}{2}$$

as desired.

c. For the sake of contradiction assume that the sequence isn't decreasing ($s_{n+1} > s_n$):

$$\frac{1}{3}(s_n + 1) > s_n$$

$$\implies \frac{s_n}{3} + \frac{1}{3} > s_n \implies 2s_n < 1 \implies s_n < \frac{1}{2}$$

as desired.

d. The limit exists because the sequence is decreasing and bounded below ($\frac{1}{2}$). Let s denote the limit.

$$s = \frac{1}{3}(s + 1) \implies s = \frac{1}{2}$$

Given that $t_1 = 1$ and $t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n$:

a. We'd like to show that the limit exists. Firstly, we can see that 0 is a lower bound for (t_n). Inductively $t_1 > 0$, and if we suppose that $t_n > 0$, then since $1 - \frac{1}{4n^2} > 0$ for all positive integer n, then t_{n+1} must also be greater than 0. Likewise, we can show that it's upper bounded by 1. Inductively, $t_1 \leq 1$, and if we suppose that $t_n \leq 1$, we have that

$$t_{n+1} \leq \left(1 - \frac{1}{4n^2}\right) \leq 1$$

because $4n^2 > 1$ for positive integer n.

Next, we will show that the sequence is also decreasing. Suppose that $t_{n+1} > t_n$ for contradiction. Then we have that:

$$\left(1 - \frac{1}{4n^2}\right) t_n > t_n$$
$$\implies \left(1 - \frac{1}{4n^2}\right) > 1$$
$$\implies 4n^2 < 1$$

which is impossible since for positive integer n, we have that $4n^2 > 1$. Since t_n is a bounded decreasing sequence, the limit exists.

b. We'd like to somehow guess the limit. We can rearrange our recurrence to get

$$t_{n+1} = \frac{4n^2 - 1}{4n^2} t_n = \frac{(2n+1)(2n-1)}{4n^2} t_n = \frac{(2n+1)(2n-1)(2n-1)(2n-3)}{4^2 n^2 (n-1)^2} t_{n-1}$$

We can extrapolate this pattern to get:

$$t_{n+1} = \frac{(2n+1)(2n-1)^2 ... (3)^2 (1)}{4^n (n!)^2}$$

The numerator can be expressed as

$$\frac{(2n+1)!(2n-1)!}{2^n n! 2^{n-1} (n-1)!} = \frac{(2n+1)!(2n-1)!}{2^{2n-1} n! (n-1)!}$$

$$t_{n+1} = \frac{(2n+1)!(2n-1)!}{2^{4n-1} (n!)^3 (n-1)!}$$

I'm not sure how to evaluate this limit, or whether it can be simplified further, but for what its worth:

$$t_6 = \frac{231231}{327680}$$

so the limit might be $\frac{2}{3}$ or something

Squeeze Test

We're given that $(a_n), (b_n), (c_n)$ are sequences such that $a_n \leq b_n \leq c_n$. We'd like to show that $\lim a_n = \lim c_n = L \implies \lim b_n = L$. We know that given any $\epsilon > 0$, there exist M, N such that $\forall n > N : a_n > L - \epsilon$ and $\forall m > M : c_m < L + \epsilon$. Then for any $n > \max(M, N)$:

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$$

$$-\epsilon < b_n - L < \epsilon$$

$$|b_n - L| < \epsilon$$

We conclude that $\lim b_n = L$