Ross 10.6

a. We'd like to show that the following sequence (s_n) is a Cauchy sequence and therefore convergent. For all n, we have that:

$$|s_{n+1} - s_n| \leq \frac{1}{2^n}$$

Fix some $\epsilon > 0$. Also lets, consider some integers m, n, and without loss of generality let $m > nn$. Then:

$$|s_m - s_n| \leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + ... + |s_{n+1} - s_n|$$

We basically turn $s_m - s_n$ into a telescoping sum and then use the triangle inequality on each individual piece. Then we use the given property to see that:

$$|s_m - s_n| \leq \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + ... + \frac{1}{2^n} < \frac{1}{2^n}\sum_{i=0}^{\infty}\frac{1}{2^i} = \frac{1}{2^{n-1}}$$

So we have that $|s_m - s_n| < \frac{1}{2^{n-1}}$, which means that as long as we pick some N such that $2^N > \frac{1}{\epsilon}$, then we're guaranteed that for $m, n > N$, we have that:

$$|s_m - s_n| < \epsilon$$

b. The result in (a) isn't true if we have that $|s_{n+1} - s_n| \leq \frac{1}{n}$. In part (a), we used the fact that

$$\sum_{i=0}^{\infty}\frac{1}{2^i}$$

converges to a particular value, which we could just account for when choosing N. Since

$$\sum_{i=0}^{\infty}\frac{1}{i}$$

diverges to infinity, then we can't say that any two points in our sequence are within an ϵ neighborhood of eachother (the distance will just keep growing).

Ross 11.2

We're given the following sequences

$$a_n = (-1)^n \qquad b_n = \frac{1}{n} \qquad c_n = n^2 \qquad d_n = \frac{6n+4}{7n-3}$$

a. For each sequence, we'd like to find a monotonically increasing subsequence

(a) The subsequence of (a_n): $(1, 1, 1, 1, 1, ...)$ of even indices is a monotonic (constant) sequence.

(b) (b_n) is already a monotone decreasing subsequence

(c) (c_n) is already a monotone increasing subsequence

(d) (d_n) is already a monotone decreasing subsequence

b. For each sequence, we'd like to give the set of subsequential limits

(a) $S_{(a_n)} = \{-1, +1\}$

(b) $S_{(b_n)} = \{0\}$

(c) $S_{(c_n)} = \{+\infty\}$

(d) $S_{(d_n)} = \{\frac{6}{7}\}$

c. For each sequence, we'd like to give the limsup and liminf

(a) $\text{limsup} s_n = 1$, $\text{liminf} s_n = -1$

(b) $\text{limsup} s_n = 0$, $\text{liminf} s_n = 0$

(c) $\text{limsup} s_n = +\infty$, $\text{liminf} s_n = +\infty$

(d) $\text{limsup} s_n = \frac{6}{7}$, $\text{liminf} s_n = \frac{6}{7}$

d. We'd like to determine which sequences converge/diverge

(a) (a_n) doesn't converge or diverge

(b) (b_n) converges to 0

(c) (c_n) diverges to $+\infty$

(d) (d_n) diverges to $\frac{6}{7}$

e. We'd like to determine which sequences are bounded

(a) (a_n) is bounded by the interval $[-1, 1]$

(b) (b_n) is bounded by the interval $[0, 1]$

(c) (c_n) is bounded below by 0 and not bounded above

(d) (d_n) is bounded (not tightly) by the interval $[0, 3]$

Ross 11.3

We're given the following sequences

$$s_n = \cos\left(\frac{n\pi}{3}\right) \qquad t_n = \frac{3}{4n+1} \qquad u_n = \left(-\frac{1}{2}\right)^n \qquad v_n = (-1)^n + \frac{1}{n}$$

a. For each sequence, we'd like to find a monotonically increasing subsequence

   (a) The subsequence of (s_n): (1, 1, 1, 1, 1, ...) of indices (0, 6, 12, ...) is a monotonic (constant) sequence.

   (b) (t_n) is already a monotone decreasing subsequence

   (c) The subsequence of (u_n): (1, $\frac{1}{4}$, $\frac{1}{16}$, ...) of even indices is a monotonic decreasing sequence.

   (d) The subsequence of (v_n): (1+$\frac{1}{2}$, 1+$\frac{1}{4}$, 1+$\frac{1}{6}$, ...) of even indices is a monotonic increasing sequence.

b. For each sequence, we'd like to give the set of subsequential limits

   (a) $S_{(s_n)} = \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$

   (b) $S_{(t_n)} = \{0\}$

   (c) $S_{(u_n)} = \{0\}$

   (d) $S_{(v_n)} = \{-1, 1\}$

c. For each sequence, we'd like to give the limsup and liminf

   (a) limsup$s_n = 1$, liminf$s_n = -1$

   (b) limsup$t_n = 0$, liminf$t_n = 0$

   (c) limsup$u_n = 0$, liminf$u_n = 0$

   (d) limsup$v_n = -1$, liminf$v_n = 1$

d. We'd like to determine which sequences converge/diverge

   (a) (s_n) doesn't converge or diverge

   (b) (t_n) converges to 0

   (c) (u_n) converges to 0

   (d) (v_n) doesn't converge or diverge

e. We'd like to determine which sequences are bounded

   (a) (s_n) is bounded by the interval $[-1, 1]$

   (b) (t_n) is bounded by the interval $[0, \frac{3}{5}]$

   (c) (u_n) is bounded (not tightly) by the interval $[-1, 1]$

   (d) (v_n) is bounded (not tightly) by the interval $[0, \frac{3}{2}]$

Ross 11.5

Given (q_n) to be an enumeration of the rationals in the interval $(0, 1]$:

a. Every real number in the interval $(0, 1]$ is a subsequential limit in this sequence. This is how we define real numbers from rational numbers, because we can take a subsequence of the enumeration that gets within any arbitrary $\epsilon-$neighborhood of some real number (as a consequence of the denseness of $\mathbb{Q}$

b. The limsup should just be 1 and the liminf should be 0, because given any enumeration, the supremum of any subsequence tail is capped at 1, and the infimum of any subsequence tail is bottomed out at 0.

Some limsup questions

I think that the best description of limsup is "the upper envelope" as the sequence goes onwards. The difference between limsup and limit is that the limit may not be well defined, but the limsup should almost always be defined. The difference between limsup and sup is that the sup could account for the beginning of the sequence (whereas limsup accounts for the tails). For example, say we have a sequence that converges to 0, but the first value is 1000. Then limsup$= 0$, but sup$= 1000$.