Ross 12.10

We'd like to show that (s_n) is bounded if and only if $\text{limsup}|s_n| < +\infty$. The first direction is kind of trivial; if $|s_n| < M$ for all n, then $\sup\{|s_N||n \in \mathbb{N}\} \leq M$, which means that $\text{limsup}|s_n|$ is definitely less than $M < +\infty$.

In the other direction, we have that $\text{limsup}|s_n| < +\infty$. Say that $\text{limsup}|s_n| = r$, and then fix some $\epsilon > 0$. We know that there's an N such that $\sup\{|s_n||n > N\} < r + \epsilon$, which means that we can just let $M = \max\{|s_1|, |s_2|, ..., |s_N|, r + \epsilon\}$, and we know that for all n, $s_n < M$ by construction.

Ross 12.12

We'd like to show that $\text{liminf} s_n \leq \text{liminf } \sigma_n \leq \text{limsup}\sigma_n \leq \text{limsup } s_n$. The middle inequality is obvious, so we can just prove the third inequality, and the first one should be a mirrored proof. So lets say that we start with $M > N$, we'd like to show that

$$\sup\{\sigma_n|n > M\} \leq \frac{1}{M}(s_1 + ... + s_N) + \sup\{s_n|n > N\}$$

$$\sup\left\{\frac{s_1 + ... + s_n}{n}|n > M\right\} = \sup\left\{\frac{s_1 + ... + s_N}{n} + \frac{s_{N+1} + ... + s_M}{n}|n > M\right\}$$

Because $n > M$, we know that:

$$\frac{s_1 + ... + s_N}{n} < \frac{s_1 + ... + s_N}{M}$$

$$\sup\left\{\frac{s_1 + ... + s_N}{M} + \frac{s_{N+1} + ... + s_M}{n}|n > M\right\} = \frac{s_1 + ... + s_N}{M} + \sup\left\{\frac{s_{N+1} + ... + s_M}{n}|n > M\right\}$$

$$\leq \frac{s_1 + ... + s_N}{M} + \sup\{s_n|n > N\}$$

as desired. Since $N > M$, we can actually write this as:

$$\leq \frac{s_1 + ... + s_N}{M} + \sup\{s_n|n > M\}$$

which gives us that the limsup $\sigma_n \leq \text{limsup} s_n$.

Ross 14.2

(a) Diverges, because we can bound the series by $\sum \frac{1}{n}$ and $\sum \frac{2}{n}$

(b) $\sum(-1)^n$ doesn't converge or diverge to $\pm\infty$ since it's bounded but not convergent

(c) $\sum \frac{3n}{n^3} = \sum \frac{3}{n^2} = 3 \sum \frac{1}{n^2}$ which converges.

(d) $\sum \frac{n^3}{3^n}$ converges since $\left|\frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}}\right| = \left|\frac{(n+1)^3}{3n^3}\right|$, which is $\frac{1}{3}$ in the limit.

(e) This converges because $n! < n^2(n-2)!$, and the numerator and denominator cancel to a convergent series.

(f) Converges because every term after 2 is less than $\frac{1}{2^n}$ which converges.

(g) Converges by ratio test, the limit should be $\frac{1}{2}$

Ross 14.10

We want to find a series that diverges by the root test, but where the ratio test gives us no information. Let

$$\sum 3^{(-1)^n - n}$$

The ratio test fails because ratios between consecutive terms change depending on the parity of the n terms vs the $n+1$ term

Rudin 3.6

a. We can look at the partial sums of this series. Each partial sum is a telescoping sum, which gives us that $\sum_1^n a_k = \sqrt{n+1} - 1$, and since the limit of the partial sums determines the convergence of the series, we conclude that this series diverges.

b. We can multiply each term a_n by $\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$, and we get that $a_n = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \leq \frac{1}{n^2}$, and since $\sum \frac{1}{n^2}$ converges $\sum a_n$ converges

c. We use the root test on this series, so the root is going to be $|a_n|^{\frac{1}{n}} = |n^{\frac{1}{n}} - 1|$. We know that the first term is converging to 1, which means that the entire term is converging to 0, which means that the series converges by the root test.

d. z is complex, which means we can probably just use $|z|$. We'd like the terms to get smaller by the sanity test, so we at least need to have $|z| > 1$. Then we can lower bound this by the geometric series $\sum \left(\frac{1}{2z}\right)^n$, and upper bound it by $\sum \left(\frac{1}{z}\right)^n$, both of which converge if $|z| > 1$.

Rudin 3.7

We know that if we have a convergent series $\sum b_n$ and a bounded series $\sum a_n$, then the product $\sum a_n b_n$ converges. So we're given that $\sum a_n$ is convergent, so $\sum \sqrt{a_n}$ is convergent, and $b_n = \frac{1}{n}$ can be bounded between 0 and 1. Therefore, the product converges.

The other two Rudin problems were pretty difficult, and given the midterm this week I didn't get as much time as I'd like to complete them.