Ross 13.3

Define B to be the set of bounded sequences $\mathbf{x} = (x_1, x_2, ...)$, and define

$$d(\mathbf{x}, \mathbf{y}) = \sup\{|x_j - y_j| : j = 1, 2, ...\}$$

a. We'd like to show that d is a metric for B. $d(\mathbf{x}, \mathbf{x}) = 0$ is clearly true.

   i. Suppose that $d(\mathbf{x}, \mathbf{y}) = 0$, in which case we must have that $x_j = y_j$ for all j, and therefore $\mathbf{x} = \mathbf{y}$.

   ii. $d(\mathbf{x}, \mathbf{y}) = \sup\{|x_j - y_j| : j = 1, 2, ...\} = \sup\{|y_j - x_j| : j = 1, 2, ...\} = d(\mathbf{y}, \mathbf{x})$

   iii. We will finally show that d satisfies the triangle inequality

$$d(\mathbf{x}, \mathbf{z}) = \sup\{|x_j - z_j| : j = 1, 2, ...\}$$

$$= \sup\{|x_j - y_j + y_j - z_j| : j = 1, 2, ...\}$$

$$\leq \sup\{|x_j - y_j| + |y_j - z_j| : j = 1, 2, ...\}$$

$$\leq \sup\{|x_j - y_j| : j = 1, 2, ...\} + \sup\{|y_j - z_j| : j = 1, 2, ...\} = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$$

b. $d*$ satisfies the properties we want, although it isn't a real valued function. It could be defined on the extended reals, but if we consider something like $d*((1, 2, 3, ...), (0, 0, 0, ...))$, we will get $+\infty$, which isn't real valued.

Ross 13.5

a. We'd first like to verify one of De Morgan's laws for sets.

$$\bigcap\{S \setminus U : U \in \mathcal{U}\} = S \setminus \bigcup\{U : U \in \mathcal{U}\}$$

Suppose $x \in \bigcap\{S \setminus U : U \in \mathcal{U}\}$. Then we know that $x \in S$ and that $x \notin U$ for all $U \in \mathcal{U}$. Since $x \notin U$ for all $U \in \mathcal{U}$, x is also not in $\bigcup\{U : U \in \mathcal{U}\}$, and therefore $x \in S \setminus \bigcup\{U : U \in \mathcal{U}\}$. The reverse inclusion is exactly the same, since we know that $x \notin \bigcup\{U : U \in \mathcal{U}\}$, and therefore $x \in S \setminus U$ for all $U \in \mathcal{U}$, so it's in their intersection. We conclude

$$\bigcap\{S \setminus U : U \in \mathcal{U}\} = S \setminus \bigcup\{U : U \in \mathcal{U}\}$$

b. Now we'd like to show that the intersection of closed sets is closed. Let $\mathcal{C}$ be a collection of closed sets. Then for every $C \in \mathcal{C}$, we know that the complement $X \setminus C$ is open. Call this open complement U_C, and let $\mathcal{U}$ denote the collection of open complements of sets in $\mathcal{C}$. Then:

$$\bigcap \mathcal{C} = \bigcap\{X \setminus U_C : U_C \in \mathcal{U}\} = X \setminus \bigcup\{U_C : U_C \in \mathcal{U}\}$$

We know that the union of open sets is open, and therefore $\bigcup\{U_C : U_C \in \mathcal{U}\}$ is open, and therefore $X \setminus \bigcup\{U_C : U_C \in \mathcal{U}\}$ is closed. Thus $\bigcap \mathcal{C}$ is closed.

Ross 13.7

We'd like to show that every open set in $\mathbb{R}$ is the disjoint union of a finite or infinite set of open intervals. Let A be an arbitrary open set. Then $\forall x \in A$, we know that there is a radius r such that $B_r(x) \subseteq A$. For each point, take the largest such radius r. We know that the union of all of these sets is definitely A. In order to make them discrete, take any two intervals that could be in our sequence, and if they aren't disjoint, then replace them with their union. Since we're allowed infinite sequences, this process doesn't necessarily have to terminate or anything, so we can safely say that we found a collection of disjoint open intervals whose union is A.

Question 4

We'd like to show that $\bar{\bar{S}} = \bar{S}$. Since $S \subseteq \bar{S}$, we must have that $\bar{S} \subseteq \bar{\bar{S}}$, so this direction of inclusion is done. Now suppose $x \in \bar{\bar{S}}$. Then there is a subsequence $(x)_n \in$
$barS$ converging to x. For each x_i, we must have a subsequence $(x_i)_j$ in S converging to x_i. Therefore, we can use a diagonalization argument to get a new subsequence in S converging to X:

$$x_{1,1}, x_{1,2}, x_{1,3}, ... \to x_1$$

$$x_{2,1}, x_{2,2}, x_{2,3}, ... \to x_2$$

$$\vdots$$

$$x_{n,1}, x_{n,2}, x_{n,3}, ... \to x_n$$

$$\vdots$$

Since there's a subsequence in S convering to X, then $x \in \bar{S}$, so $\bar{\bar{S}} \subseteq \bar{S}$, and both inclusions give us $\bar{\bar{S}} = \bar{S}$

Question 5

We'd like to show that $\bar{S}$ is the intersection of all closed subsets in X containing S. Suppose that A is a closed subset of X containing S. Then A contains S, as well as all of it's subsequential limit points (A is closed), which means that $\bar{S} \subseteq A$. Let $S*$ be the intersection of all closed subsets of X containing S. We know from above that $\bar{S} \subseteq S*$, but also, since $\bar{S}$ is a closed subset of X containing S, we know that $S* \subseteq \bar{S}$, and therefore $S* = \bar{S}$, and $\bar{S}$ is the intersection of all closed subsets of X containing S.