1

We'd like to prove that $[0,1]^2 \subseteq \mathbb{R}^2$ is sequentially compact. So let (x_n, y_n) be a sequence of points in the unit square. We know that the sequence of x coordinates, $(x)_n$ is a bounded sequence, and therefore by the Bolzano Weierstrauss theorem, it has a convergent subsequence $(x_n)_k$ (suppose that this converges to value p). Now we also know that the sequence of y coordinates with the same indices $(y_n)_k$ is also a bounded sequence, and therefore it has a convergent subsequence $(y_{n_k})_m$ (that converges to value q). Since $(x_n)_k$ converges to p, $(x_{n_k})_m$ also converges to p, and therefore the subsequence $(x_{n_{k_m}}, y_{n_{k_m}})$ in $[0,1]^2$ converges to (p, q).

This generalizes, even though Bolzano Weierstrauss doesn't always generalize, because if X, Y are sequentially compact, then already know that every sequence has a convergent subsequence, and we can extend our indexing argument to $X \times Y$

2

We'd like to show that the set of points in $[0,1]$, whose decimal expansions consist only of 4 and 7, is uncountable and compact. Firstly, to show it's uncountable, we can use diagonalization. Suppose we have an enumeration of every real number whose decimal expansions consist of only 4 and 7. Construct a new real number as follows:

To construct the i^{th} decimal digit of our new real number r, we take the i^{th} digit of the i^{th} number in our enumeration, and if it's a 4, our digit will be a 7, and if it's a 7, our digit will be a 4. Therefore we've constructed a real number consisting of decimal digits only 4 and 7 that cannot be in the image of our enumeration.

This set is compact. Firstly, we know that it's bounded by $[0,1]$ (and more specifically between $[0.\overline{44}, 0.\overline{77}]$). Since we're working in the real numbers, every sequence has a convergent subsequence, and we know that the convergent subsequence must have a limit in the interval $[0.\overline{44}, 0.\overline{77}]$, with digits only consisting of 4 and 7. We also know that both endpoints of the interval are contained in the set, and therefore we know that the set is closed. Since our set is closed and bounded, it must be sequentially compact.

3

Let $A_1, A_2, \cdots$ be subset of a metric space. We want to show that it is possible that if $B = \cup_i A_i$, then $\bar{B} \supset \cup_i \bar{A}_i$ can be a strict inclusion. Consider an enumeration of the rational numbers, and let A_i be the singleton containing the i^{th} rational number. Then $B = \cup_i A_i = \mathbb{Q}$. Since finite sets are closed, $\bar{A}_i = A_i$, and we know that $\bar{B} = \bar{\mathbb{Q}} = \mathbb{R}$.

$$\mathbb{R} = \bar{B} \supset \cup_i \bar{A}_i = \cup_i A_i = \mathbb{Q}$$

is definitely a strict inclusion.

4

This argument fails because it isn't necessarily true that a countably infinite number of closed sets intersect at a closed set. For example, take the countable union of the sets A_i, where A_i is the singleton containing $\{\frac{1}{i}\}$. This set doesn't contain one of it's subsequential limits (0), and therefore cannot be closed. To find a closed set that can't be written as a countable union of closed sets, we can just use the set from problem 2. Take any 2 elements of this set. There isn't a continuous interval that contains both elements and no elements that have digits other than 4 or 7, which means that that set is a union of uncountably many singletons (so it's not a countable union).