1

We'd like to show that if X and Y are open cover compact, then $X \times Y$ is open cover compact. Consider any open cover of $X \times Y$, call it O. This is a set of open sets containing elements of $X \times Y$. We can project O onto X by considering the projection of each open set in $X \times Y$ onto X, which means that this new set (call it O_X) is a set containing open sets in X such that it contains every $X-$coordinate represented in $X \times Y$. Similarly define O_Y. Since O_X is an open cover of X, and X is open cover compact, it has a finite subcover O'_X. Similarly, O_Y has an open subcover O'_Y in Y. Therefore we can consider the following set:

$$O' = \{A \times B | A \in O'_X, B \in O'_Y\}$$

Since each A, B is open, we know that the product is open, which means that O' is a set of open sets. Furthermore, consider any point $(x, y) \in X \times Y$. We know that x is contained in one of the sets in our finite subcover of X, and y is contained in one of the sets in our finite subcover of Y, therefore the point (x, y) is contained inside one of the open Cartesian products in O'. Further, we conclude that since O'_X and O'_Y are both finite subcovers, the set of pairs of Cartesian products is necessarily finite, (since $|O'| = |O'_X \times O'_Y|$). We conclude that O has a finite subcover.

2

We want to determine whether the following statements are true or false for a continuous function f.

1. If A is open, then $f(A)$ is open. This is false. Let $f(x) = x^2$ and let $A = (-1, 1)$. $f(A)$ isn't open because there is no open ball around 0.

2. If A is closed, then $f(A)$ is closed. This is also false. Consider the function $f(x) = \arctan(x)$ with domain $\mathbb{R}$ (which is closed over itself) and image $f(A) = (-\frac{\pi}{2}, \frac{\pi}{2})$. The image does not contain it's limit points, and thus isn't closed.

3. If A is bounded, then $f(A)$ is bounded. This is also false. Consider the function:

   $$\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$$

   The function is continuous on the domain, the domain is bounded (by $-\frac{\pi}{2}, \frac{\pi}{2}$), but the codomain is unbounded.

4. If A is compact, then $f(A)$ is compact. This is true, since if we consider an open cover of $f(A)$, then we can consider the preimage of each cover, which we know has a finite subcover in A, and we can just take the set's whose preimage maps to the sets in the finite subcover of A.

5. If A is compact, then $f(A)$ is connected. We can prove the contrapositive. Suppose that $f(A)$ isn't connected. Then it can be written as $f(A) = U \sqcup V$, which means that $A = f^{-1}(U) \sqcup f^{-1}(V$, which means that A isn't connected (since the preimage of open sets are open by continuity of f).

3

Consider a sequence $(1, 2, 3, ...)$ in the codomain $\mathbb{R}$, and suppose such a function f exists. Since f is surjective, we know that we can choose an element of the preimage of each sequence member to get a sequence $(x_1, x_2, ...)$ in the domain $[0, 1]$. Since $[0, 1]$ is compact, this has a convergent subsequence, and since f is continuous, we know that it preserves the convergence of sequences. However, $(1, 2, 3, ...)$ has no convergent subsequence in $\mathbb{R}$, we've reached a contradiction.