1

Let $f_n(x) = \frac{n+\sin x}{2n+\cos n^2 x}$. We want to show that this converges uniformely on $\mathbb{R}$. We can lower bound and upper bound sin and cos by -1 and 1 respectively. We'll see that:

$$\frac{n-1}{2n+1} \leq f_n(x) \leq \frac{n+1}{2n-1}$$

Looking at this, we expect $f_n(x)$ to converge to $\frac{1}{2}$. Firstly, note that $-\frac{n-1}{2n-1} \leq \frac{n-1}{2n+1}$. To verify this, cross multiply both sides and we'll see that that inequality is equivalent to $n^2 \geq n$ which is true. Therefore, we have that:

$$\left|f_n(x) - \frac{1}{2}\right| \leq \frac{n+1}{2n-1} - \frac{1}{2} = \frac{3}{4n-2}$$

Fix some $\epsilon > 0$. We want $\frac{3}{4N-2} < \epsilon$. We can just choose N such that $N > \frac{3+2\epsilon}{4\epsilon}$, and then we meet the uniform convergence criteria as desired.

2

We're given that $f(x) = \sum_{n=1}^{\infty} a_n x^n$, and we want to show that it's continuous on $[-1,1]$ if $\sum_n |a_n| < \infty$. Given that $x \in [-1,1]$, we know that $-1 \leq x^n \leq 1$ for all n. Therefore, $|a_n x^n| \leq |a_n|$, which we know converges. Firstly, note that every polynomial is continuous, which means that every partial sum is continuous. By the Weierstrauss M test, the sum uniformly converges, and since f is a limit of uniformely convergent continuous functions, it is continuous.

3

We want to show that $f(x) = \sum_n x^n$ is continuous on $(-1,1)$ but convergence is not uniform. We'll use the hint to show that we have uniform convergence on $[-a,a]$ for $0 < a < 1$. Firstly, for $a \in (0,1)$, $[-a,a] \subseteq (-1,1)$. Choose some arbitrary $p \in (-1,1)$. We know that there's some $a \in (0,1)$ such that $p \in [-a,a]$. We know that since $p \in [-a,a]$, $|p| \leq |a|$, and therefore, $|p^n| \leq |a|^n$, and by the Weierstrauss M test this converges (it's a geometric series with ratio less than 1). So f uniformely converges on all intervals $[-a,a]$ for $a \in (0,1)$.

Now we want to show it doesn't converge uniformely on $(-1,1)$. Firstly note that partial sums of f have closed form: $f_n(x) = \frac{x(1-x^n)}{1-x}$, and $f = \frac{x}{1-x}$ (using our formulas for geometric series, where since we are 1-indexing the first term is x). For arbitrary n:

$$f_n(x) - f(x) = x\frac{1-x^n-1}{1-x} = \frac{x^{n+1}}{1+x}$$

If we fix some $\epsilon > 0$, we can always make $|f_n(x) - f(x)|$ greater than ϵ by setting x very close to 1 (say within $(1-\epsilon^n, n)$).