1

Let $f_n(x) = \frac{n + \sin x}{2n + \cos n^2 x}$. We want to show that this converges uniformely on \mathbb{R} . We can lower bound and upper bound sin and cos by -1 and 1 respectively. We'll see that:

$$\frac{n-1}{2n+1} \le f_n(x) \le \frac{n+1}{2n-1}$$

Looking at this, we expect $f_n(x)$ to converge to $\frac{1}{2}$. Firstly, note that $-\frac{n-1}{2n-1} \leq \frac{n-1}{2n+1}$. To verify this, cross multiply both sides and we'll see that that inequality is equivalent to $n^2 \geq n$ which is true. Therefore, we have that:

$$\left| f_n(x) - \frac{1}{2} \right| \le \frac{n+1}{2n-1} - \frac{1}{2} = \frac{3}{4n-2}$$

Fix some $\epsilon > 0$. We want $\frac{3}{4N-2} < \epsilon$. We can just choose N such that $N > \frac{3+2\epsilon}{4\epsilon}$, and then we meet the uniform convergence criteria as desired.

$\mathbf{2}$

We're given that $f(x) = \sum_{n=1}^{\infty} a_n x^n$, and we want to show that it's continuous on [-1, 1] if $\sum_n |a_n| < \infty$. Given that $x \in [-1, 1]$, we know that $-1 \leq x^n \leq 1$ for all n. Therefore, $|a_n x^n| \leq |a_n|$, which we know converges. Firstly, note that every polynomial is continuous, which means that every partial sum is continuous. By the Weierstrauss M test, the sum uniformly converges, and since f is a limit of uniformly convergent continuous functions, it is continuous.

3

We want to show that $f(x) = \sum_{n} x^{n}$ is continuous on (-1, 1) but convergence is not uniform. We'll use the hint to show that we have uniform convergence on [-a, a] for 0 < a < 1. Firstly, for $a \in (0, 1)$, $[-a, a] \subseteq (-1, 1)$. Choose some arbitrary $p \in (-1, 1)$. We know that there's some $a \in (0, 1)$ such that $p \in [-a, a]$. We know that since $p \in [-a, a]$, $|p| \leq |a|$, and therefore, $|p^{n}| \leq |a|^{n}$, and by the Weierstrauss M test this converges (it's a geometric series with ratio less than 1). So f uniformely converges on all intervals [-a, a] for $a \in (0, 1)$.

Now we want to show it doesn't converge uniformely on (-1, 1). Firstly note that partial sums of f have closed form: $f_n(x) = \frac{x(1-x^n)}{1-x}$, and $f = \frac{x}{1-x}$ (using our formulas for geometric series, where since we are 1-indexing the first term is x). For arbitrary n:

$$f_n(x) - f(x) = x \frac{1 - x^n - 1}{1 - x} = \frac{x^{n+1}}{1 + x}$$

If we fix some $\epsilon > 0$, we can always make $|f_n(x) - f(x)|$ greater than ϵ by setting x very close to 1 (say within $(1 - \epsilon^n, n)$).