1

We want to construct a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $x \leq 0$, $f(x) = 1$ for $x \geq 1$, and for $x \in (0,1)$, $f(x) \in [0,1]$. Rudin gives:

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x}} & x > 0 \end{cases}$$

We know $f(0) \to 0$ and f is infinitely differentiable at 0. Then we have that the function:

$$\frac{f(x)}{f(x) + f(1-x)}$$

Should be smooth since f is smooth, and when we evaluate it at the endpoints, we get 0 at 0 and 1 at 1. We can just let the other parts be piecewise defined, and since we know that f nicely converges to 0, as x goes to 0, this function should be smooth at it's endpoints too.

2

We can define:

$$f = c_0 x + \frac{1}{2} c_1 x^2 + ... + \frac{1}{n+1} c_n x^{n+1}$$

We see that $f(0) = 0$ and by assumption $f(1) = 0$. By Rolle's theorem, there is a point $c \in [0,1]$ such that $f'(c) = 0$. f' is actually the function we are looking for:

$$f' = c_0 + ... + c_n x^n$$

so we have shown the existence of a c such that $f'(c) = 0$

3

We have that f' is continous on $[a,b]$ and $\epsilon > 0$. Let $g(x) = x$. Then by the mean value theorem, there is $c \in [t,x]$ such that:

$$[f(t) - f(x)] = (t-x)f'(c)$$

and thus we have that the difference quotient is $f'(c)$. We know that there is $\delta > 0$ such that $|c - x| < \delta$, we have $|f'(c) - f'(x)| < \epsilon$.

4

We define $Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$. We will differentiate the expression:

$$Q(t)(t-\beta) = f(t) - f(\beta)$$

using the product rule on the LHS to get:

$$f^{(n-1)}(t) = (n-1)Q^{(n-2)}) + (t-\beta)Q^{(n-1)}$$

We plug this into our original expression for the Taylor expansion:

$$P_\alpha(\beta) = \sum_{i=i}^{n-1} \frac{f^{(i)}(\alpha)}{i!}(\beta - \alpha)^i + f(\alpha)$$

We plut in our expression for $f^{(i)}$ to get that:

$$\begin{aligned} P_\alpha(\beta) &= \sum_{i=i}^{n-1} \frac{iQ^{(i-1)}) + (\alpha-\beta)Q^{(i)}}{i!}(\beta-\alpha)^i + f(\alpha) \\ &= \sum_{i=i}^{n-1} \frac{Q^{(i-1)}(\alpha)}{(i-1)!}(\beta-\alpha)^i - \sum_{i=i}^{n-1} \frac{Q^{(i)}(\alpha)}{i!}(\beta-\alpha)^{i+1} + f(\alpha) \end{aligned}$$

This will telescope, and we get the first term of the first sum minus the last term of the second sum:

$$P(\beta) = Q(\alpha)(\beta-\alpha) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n + f(\alpha)$$

If we plug in our expression for $Q(\alpha)$, we get what we want, that:

$$P(\beta) = f(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n$$

5

a. If $f'(t) \neq 1$ for all $t \in \mathbb{R}$, then f has at most 1 fixed point. Suppose f has 2 fixed points (a, b). Then apply the mean value theorem with $g(x) = x$, so:

$$f(a) - f(b) = (a-b)f'(c) \implies f'(c) = 1$$

contradicting our assumption.

b. We want to show $f(t) = t + (1+e^t)^{-1}$ has no fixed point. Say it does, then $t = t + (1+e^t)^{-1}$, so $(1+e^t)^{-1} = 0$, which is impossible, as the numerator is nonzero (even though the whole thing asymptotically approaches 0).

c. A fixed point is an intersection of the function with $g(x) = x$. If $|f'(x)| < 1$, f has to intersect with g, which means that a fixed point exists. We expect x_{n+1} to be close to $f(x_{n+1})$ since we are converging to the fixed point, so we want:

$$|f(x_{n+1}) - x_{n+1}| < |f(x_n) - x_n|$$
$$\iff |f(f(x_n)) - f(x_n)| < |f(x_n) - x_n|$$

Suppose not:

$$|f(f(x_n)) - f(x_n)| \geq |f(x_n) - x_n|$$

We know there is $c \in [x_n, f(x_n)]$ such that by mean value theorem, taking $g(x) = x$:

$$f(f(x_n)) - f(x_n) = (f(x_n) - x_n)f'(c)$$

But then:

$$f'(c) = \frac{f(f(x_n)) - f(x_n)}{f(x_n) - x_n} \geq 1$$

Which contradicts our assumption that $f'(t) < 1$ for all t.

d. The process is just the zigzag path since $x_{n+1} = f(x_n)$, so we get pairs $(x_1, f(x_1)) = (x_1, x_2)$, and then $(f(x_1), f(f(x_1)))) = (x_2, x_3)$ and so on.