Ross 33.4

We want to give an example of a function on $[0,1]$ that isn't integrable, but for which $|f|$ is integrable:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 * x \notin \mathbb{Q} \end{cases}$$

Since our domain is $[0,1]$ we know that for any interval in a partition of $[0,1]$ will contain a rational and irrational number, and therefore:

$$U(f) - L(f) = 1 - (-1) = 2 \neq 0$$

Ross 33.7

We're given a bounded function on $[a,b]$ where $\exists B > 0$ such that $|f(x) \leq B$ for all $x \in [a,b]$.

a. We first want to show that:

$$U(f^2, P) - L(f^2, P) \leq 2B[U(f,P) - L(f,P)]$$

Let a_i be the point in the i^{th} interval that realizes the maxmimum value of f^2, and let b_i be the point that realizes the minimum value. Then $\forall i$, we have that $f(a_i) - f(b_i) \leq \sup_{I_i} f - \inf_{I_i} f$ because the supremum could be larger and the infimum could be smaller. We also know by assumption that $f(a_i) + f(b_i) \leq 2B$. Also let Δ_i be the length of the i^{th} segment. We have:

$$\begin{aligned} U(f^2, P) - L(f^2, P) &= \sum_{i=1}^{n} (f^2(a_i) - f^2(b_i))\Delta_i \\ &= \sum_{i=1}^{n} (f(a_i) - f(b_i))(f(a_i) + f(b_i))\Delta_i \\ &\leq \sum_{i=1}^{n} 2B(\sup_{I_i} f - \inf_{I_i} f)\Delta_i \\ &= 2B[U(f,P) - L(f,P)] \end{aligned}$$

as desired

b. Now we want to show that f being integrable on $[a,b]$ implies f^2 being integrable on $[a,b]$. Let $\epsilon > 0$. Since f is integrable on $[a,b]$, there exists P such that:

$$U(f,P) - L(f,P) < \frac{\epsilon}{2B}$$

$$\implies 2B(U(f,P) - L(f,P)) < \epsilon$$

$$\implies U(f^2,P) - L(f^2,P) < \epsilon$$

where the last implication follows from part (a)

Ross 33.13

Suppose that f, g continuous on $[a, b]$ such that:

$$\int_a^b f = \int_a^b g$$

We want to show that theres $x \in [a, b]$ such that $f(x) = g(x)$. We use the intermediate value theorem: $\int_a^b f - g = 0$, which means that there exists f such that:

$$(f-g)(x) = \frac{1}{b-a}\int_a^b f - g = 0$$
$$\implies (f-g)(x) = 0 \implies f(x) = g(x)$$

Ross 35.4

Given that $F(t) = \sin(t)$ for $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we want to compute the following integrals:

a. We simplify the integral as follows:

$$\int_0^{\frac{\pi}{2}} x \, \mathrm{d}F = \int_0^{\frac{\pi}{2}} x\cos(x)\mathrm{d}x$$

Now we use integration by parts:

$$\int_0^{\frac{\pi}{2}} x\cos(x)\mathrm{d}x = \Big[x\sin(x) + \cos(x)\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$$

b. We simplify the integral as follows:

$$\int_0^{\frac{\pi}{2}} x \, \mathrm{d}F = \int_0^{\frac{\pi}{2}} x\cos(x)\mathrm{d}x$$

Now we use integration by parts:

$$\int_0^{\frac{\pi}{2}} x\cos(x)\mathrm{d}x = \Big[x\sin(x) + \cos(x)\Big]_{\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2} - \frac{\pi}{2} = 0$$

Ross 35.9a

Given that f is continuous on $[a, b]$, we want to show that there is $x \in [a, b]$ such that:

$$\int_a^b f \, \mathrm{d}F = f(x)\big[F(b) - F(a)\big]$$

Pick α, β such that $\inf f = f(\alpha), \sup f = f(\beta)$, where the supremum and infimum are taken over $[a, b]$. Then:

$$f(\alpha)\int_a^b \mathrm{d}F \le \int_a^b f \, \mathrm{d}F \le f(\beta)\int_a^b \mathrm{d}F \implies f(\alpha)\big[F(b) - F(a)\big] \le \int_a^b f \, \mathrm{d}F \le f(\beta)\big[F(b) - F(a)\big]$$

We divide by $\big[F(b) - F(a)\big]$ and apply the intermediate value theorem to get $\gamma \in [\alpha, \beta]$ such that

$$f(\gamma) = \frac{1}{F(b) - F(a)}\int_a^b f \, \mathrm{d}F$$