Ross 34.2

If $F(x) = \int_0^x e^{t^2}\ \mathrm{d}t$, then:

$$\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\ \mathrm{d}t = \lim_{x\to 0}\frac{F(x)-F(0)}{x}$$

This is just $F'(0)$, which by the fundamental theorem of calculus is $e^0 = 1$.

If $F(x) = \int_0^x e^{t^2}$ again, then:

$$\lim_{h\to 0}\frac{1}{h}\int_3^{3+h} e^{t^2} = \frac{F(3+h)-F(3)}{h} = F'(3) = e^9$$

Ross 34.5

Given continuous f, we want to show that $F(x) = \int_{x-1}^{x+1} f(t)\ \mathrm{d}t$ is differentiable and we want to find f'. We can define $h(x) = \int_0^x f(t)\mathrm{d}t$. Our goal is to show that:

$$\lim_{\epsilon\to 0}\frac{F(x+\epsilon)-F(x)}{\epsilon} = \frac{\int_{x+\epsilon-1}^{x+\epsilon+1} f(t)\ \mathrm{d}t - \int_{x-1}^{x+1} f(t)\ \mathrm{d}t}{\epsilon}$$

exists at all x. We can separate each integral and realize that:

$$\lim_{\epsilon\to 0}\frac{\int_{x+\epsilon-1}^{x+\epsilon+1} f(t)\ \mathrm{d}t - \int_{x-1}^{x+1} f(t)\ \mathrm{d}t}{\epsilon} = \lim_{\epsilon\to 0}\frac{h(x+\epsilon+1)-h(x+1)}{\epsilon} - \lim_{\epsilon\to 0}\frac{h(x+\epsilon-1)-h(x-1)}{\epsilon}$$

the first term on the RHS is just $h'(x+1) = f(x+1)$, the second term is $h'(x-1) = f(x-1)$. Therefore F' exists and $F'(x) = f(x+1) - f(x-1)$

Ross 34.7

We want to find $\int_0^1 x\sqrt{1-x^2}\ \mathrm{d}x$. Let $u = 1-x^2$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = -2x$, so:

$$\int_0^1 x\sqrt{1-x^2}\ \mathrm{d}x = \frac{1}{2}\int_0^1 \sqrt{u}\ \mathrm{d}u$$

when we evaluate the bounds, we're taking the interval from 1 to 0, so we swap the bounds and multiply by -1, which cancels with the $-\frac{1}{2}$.

$$\frac{1}{2}\int_0^1 \sqrt{u}\ \mathrm{d}u = \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^1 = \frac{1}{3}$$