Ross

Induction

1.10) If $n=1$, $2n+1 = 3 = 4n-1 = 3n^2$ ✓

If $(2n+1)+(2n+3)+\dots+(4n-1) = 3n^2$ for a given n

then WTS $(2(n+1)+1)+(2(n+1)+3)+\dots(4(n+1)-1) = 3(n+1)^2$

$$= (2n+3)+(2n+5)+\dots 4n+3$$
$$= 3n^2-(2n+1)+(4n+1)+(4n+3)$$
$$= 3n^2+6n+3$$
$$= 3(n+1)^2 \quad ✓$$

1.12) a) $(a+b)^1 = a+b = a^1+b^1$ ✓

$(a+b)^2 = a^2+2ab+b^2 = a^2+2\cdot ab+b^2$ ✓

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3 = a^3+3\cdot a^2b+3\cdot ab^2+b^3$ ✓

b)
$$\frac{n!}{K!(n-K)!} + \frac{n!}{(K-1)!(n-K+1)!}$$
$$= \frac{n!}{K!(n-K)!} + \frac{n!\,K}{K!(n-K)!(n-K+1)}$$
$$= \frac{n!(n-K+1)+n!\,K}{K!(n-K)!(n-K+1)}$$
$$= \frac{n!(n+1)}{K!(n-K+1)!} = \binom{n+1}{K}$$

c) We already verified base case for n in part a

If it's true for n, $(a+b)^{n+1} = (a+b)^n(a+b)$

$$= \left(\binom{n}{0}a^n+\dots+\binom{n}{n}b^n\right)(a+b)$$
$$= \binom{n+1}{0}a^{n+1} + \left(\binom{n}{K}+\binom{n}{K+1}\right)a^{n-K}b^{K+1} + \dots + \binom{n+1}{n+1}b^{n+1}$$
$$= \sum_{K=0}^{K=n+1}\binom{n+1}{K}a^{n-K+1}b^K$$

$\therefore$ Binomial Thm is true

2.1) Assume $\sqrt{x} = \frac{p}{q}$ For some $p, q \in \mathbb{Z}$ coprime

then $p^2 = x q^2$

If q is divisible by x then p is not, bcs coprime but $p = xq^2$ and $q \in \mathbb{Z}$ so contradiction.

So q is not divisible by x. But since $x | p^2$, x must divide q, which is not the case or x must be square.

None of $x = 3, 5, 7, 24, 31$ are square,

so we have a contradiction.

2.2) Assume $x^{\frac{1}{y}} = \frac{p}{q}$ for some coprime $p, q \in \mathbb{Z}$.

then $p^y = x q^y$

By reasoning above, if q is divisible by x, we have contradiction since coprime.

Since $x | p^y$, x must divide q or $x^{\frac{1}{y}} \in \mathbb{Z}$

but we know $\sqrt[3]{2}$, $\sqrt[7]{5}$ & $\sqrt[4]{13}$ are not $\mathbb{Z}$ (duh) so they can't be rational either.

2.7) a) $\sqrt{4+2\sqrt{3}} - \sqrt{3} \cdot \dfrac{\sqrt{4+2\sqrt{3}} + \sqrt{3}}{\sqrt{4+2\sqrt{3}} + \sqrt{3}}$

$= \dfrac{4+2\sqrt{3} - 3}{\sqrt{4+2\sqrt{3}} + \sqrt{3}}$

2.7a) $\left(\sqrt{4+2\sqrt{3}}-\sqrt{3}\right)^2 = 4+2\sqrt{3}+3-2\sqrt{12+6\sqrt{3}}$

$\sqrt{4+2\sqrt{3}+3-3}-\sqrt{3}$

$\sqrt{(1+\sqrt{3})^2}-\sqrt{3}$

$\boxed{1+\sqrt{3}-\sqrt{3}}=\boxed{1}$

b) $\sqrt{6+4\sqrt{2}}-\sqrt{2}$

$\sqrt{6+4\sqrt{2}+2-2}-\sqrt{2}$

$\sqrt{4(1+\sqrt{2}+\frac{1}{2})}-\sqrt{2}$

$2\sqrt{\left(1+\frac{1}{\sqrt{2}}\right)^2}-\sqrt{2}$

$2\left(1+\frac{1}{\sqrt{2}}\right)-\sqrt{2}$

$2+\sqrt{2}-\sqrt{2}=\boxed{2}$

3.6 a) $|a+b+c| \le |a+b| + |c|$ by triangle ineq
$\le |a|+|b|+|c|$

b) If $|a_1+a_2+\dots+a_n| \le |a_1|+|a_2|+\dots|a_n|$ } base case above and triangle

$|a_1+a_2+\dots+a_n+a_{n+1}| \le |a_1+a_2+\dots+a_n|+|a_{n+1}|$ by triangle
$\le |a_1|+|a_2|+\dots+|a_{n+1}|$ by inductive hypothesis

QED

4.11) $a<b$ $a,b \in \mathbb{R}$ Denseness of $\mathbb{Q}$

Base case $\exists \geq$ rational r between a, b by thm 4.7

Inductive hypothesis: $a<b$, $a,b \in \mathbb{R} \Rightarrow \exists\, n\ r_i$'s such that $a<r_i<b\ \forall r_i$

Choose $\min(\{r_i\})$ it is real so by denseness of $\mathbb{Q}$ (4.7)

$\exists\, a<r_{i+1}<r_i<b$ where $r_{i+1} \in \mathbb{Q}$ so

we used inductive hypothesis to show $n+1$ case so there must be infinitely many rationals between a & b.

4.14) a) $\forall a \in A, \forall b \in B$, $\sup(A+B) - b \geq a$

yes $\sup(A+B) \geq a+b$ by def of Sup and $a+b \in A+B$

Hence $\sup A \leq \sup(A+B) - b \quad \forall b \in B$

Then $\sup(A+B) - \sup A$ is upper bound for B

since $b \leq \sup(A+B) - \sup(A) \quad \forall b$.

$\therefore \sup(A+B) \geq \sup(A) + \sup(B)$

But also $\forall a+b \in A+B$, $a \leq \sup(A)$, $b \leq \sup(B)$

so $a+b \leq \sup(A) + \sup(B)$

so $\sup(A+B) \leq \sup(A) + \sup(B)$

$\sup(A+B) = \sup(A) + \sup(B)$

b) $-\sup(-X) = \operatorname{Inf}(X)$

$-\sup(-A-B) = -\sup(-A) - \sup(-B)$

$\operatorname{Inf}(A+B) = \operatorname{Inf}(A) + \operatorname{Inf}(B)$

7.5) a) $\lim_{n\to\infty} \sqrt{n^2+1} - n$

$\lim_{n\to\infty} n\sqrt{1+\frac{1}{n^2}} - n$

$\lim_{n\to\infty} n\sqrt{1+0} - n$

$\boxed{0}$

$S_n = \sqrt{n^2+1} - n \cdot \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$

$= \frac{1}{\sqrt{n^2+1}+n} =$

b) $\lim_{n\to\infty} \sqrt{n^2+n} - n$

$\lim_{n\to\infty} n\sqrt{1+\frac{1}{n}} - n$

$\lim_{n\to\infty} n\sqrt{1+0} - n$

$\boxed{0}$

c) $\lim_{n\to\infty} \sqrt{4n^2+n} - 2n$

$\lim_{n\to\infty} n\sqrt{4+\frac{1}{n}} - 2n$

$\lim_{n\to\infty} 2n - 2n$

$\boxed{0}$