9.9

(a) There exist N_0 s.t $s_n \leq t_n$ for $\forall n > N_0$

There exist $M > 0$, $N_1 \in \mathbb{N}$ s.t $s_n > M$ $\forall n > N_1$

Take $N = \max\{N_0, N_1\}$

$$M < s_n \leq t_n$$

$$M < t_n \quad \forall n > N$$

Therefore $\lim t_n = +\infty$.

(b) There exist N_0 s.t $s_n \leq t_n$ for $\forall n > N_0$

There exist $M > 0$, $N_1 \in \mathbb{N}$ s.t $t_n < M$ $\forall n > N_1$

Take $N = \max\{N_0, N_1\}$

$$M > t_n \geq s_n$$

$$M > s_n \quad \forall n > N$$

Therefore $\lim s_n = -\infty$

(c) Assume $t_n - s_n = A_n$ and $\lim A_n = L < 0$

$\lim A_n = L$

$\forall \varepsilon > 0$, $L + \varepsilon < 0$

There exist N_1 such that $L - \varepsilon < A_n < L + \varepsilon$ $\forall n \geq N_1$

$N = \max\{N_0, N_1\}$

$$A_n < L + \varepsilon < 0 \quad \forall n \geq N$$

$$A_n < 0 \quad \forall n \geq N$$

Which is contradiction

Therefore $\lim s_n \leq \lim t_n$

9.15 Show $\lim_{n \to \infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$

$$\lim_{n \to \infty} \left| \frac{a^n}{n!} \cdot \frac{(n+1)!}{a^{n+1}} \right|$$

$$= \lim \left| \frac{a}{n+1} \right|$$

$$= 0. < 1$$

10.7 Want to show $\lim S_n = \sup S$.

There exist u such that $u = \sup S$

$\exists n > 0$ s.t $\frac{1}{n} > 0$

$$u - \frac{1}{n} < S_n < u + \frac{1}{n}$$

$$\lim (u - \frac{1}{n}) \leq \lim S_n \leq \lim (u + \frac{1}{n})$$

$$\lim S_n = u = \sup S.$$

Therefore $\lim S_n = \sup$.

10.8. S_n is increasing sequence of positive numbers.

$$S_1 \leq S_2 \leq S_3 \leq \cdots \leq S_n \leq S_{n+1}$$

$$\frac{1}{n} (S_1 + \cdots + S_n) \leq S_n \leq S_{n+1}$$

Want to show $\sigma_{n+1} > \sigma_n$

$$S_{n+1} \geq \frac{1}{n} [(S_1 + \cdots + S_n) + n(S_1 + \cdots + S_n) - n(S_1 + \cdots + S_n)]$$

$$\geq \frac{1}{n} [(n+1)(S_1 + \cdots + S_n) - n(S_1 + \cdots + S_n)]$$

$$\geq \frac{n+1}{n} (S_1 + \cdots + S_n) - (S_1 + \cdots + S_n)$$

Therefore $\sigma_{n+1} > \sigma_n$, σ_n is increasing sequence.

10.9

Let $S_1=1$ and $S_{n+1} = (\frac{n}{n+1})S_n^2$ for $n \geq 1$

(a) find S_2, S_3 and S_4

$S_2 = S_{1+1} = (\frac{1}{1+1}) S_1^2 = (\frac{1}{1+1}) 1^2 = \frac{1}{2}$

$S_3 = S_{2+1} = (\frac{2}{2+1}) S_2^2 = \frac{2}{3} \times (\frac{1}{2})^2 = \frac{1}{6}$

$S_4 = S_{3+1} = (\frac{3}{3+1}) S_3^2 = \frac{3}{4} \times (\frac{1}{6})^2 = \frac{1}{48}$

(b) $S_1=1$, $S_2=\frac{1}{2}$, $S_3=\frac{1}{6}$, $S_4=\frac{1}{48}$

So we got S_n is decreasing.

$S_{n+1} = (\frac{n}{n+1}) S_n^2$

$< S_n^2$

$= S_n \cdot S_n$

$\leq S_n$

$= S_n$

$\because$ monotone decreasing and bounded.

$\therefore$ So it is convergent., $\lim S_n$ exists.

(c) $S_{n+1} < S_n < S_1 = 1$

$\lim S_{n+1} = \lim (\frac{n}{n+1}) S_n^2$

$= \lim (\frac{n}{n+1}) \lim (S_n^2)$

$= \lim (\frac{1}{1+\frac{1}{n}}) \lim (S_n^2)$

$= 1 \lim (S_n^2) = \lim (S_n^2)$

$\because$ S_{n+1} and S_n is convergent, $\therefore \lim (S_n^2) = \lim (S_n)$

$\lim S_n^2 - \lim S_n = 0$ we got $\lim S_n = 1$ or 0

$\because S_n < S_1 = 1$ $\therefore \lim S_n = 0$

10.10 (a)

$S_2 = S_{1+1} = \frac{1}{3}(1+1) = \frac{2}{3}$

$S_3 = S_{2+1} = \frac{1}{3}(\frac{2}{3}+1) = \frac{5}{9}$

$S_4 = S_{3+1} = \frac{1}{3}(\frac{5}{9}+1) = \frac{14}{27}$

(b) Show $S_n > \frac{1}{2}$ for all n

when $n=1$ $S_1 = 1 > \frac{1}{2}$ true

when $n = P(k+1)$ $S_{k+1} = \frac{1}{3}(S_k+1)$

$> \frac{1}{3}(\frac{1}{2}+1)$

$> \frac{1}{3} \times \frac{3}{2}$

$> \frac{1}{2}$

$\therefore S_{k+1} > \frac{1}{2}$

(C) Want to show $S_{n+1} < S_n$

From part (a), $S_1 > S_2 > S_3 > \cdots$

When $n=1$, $S_1 > S_2$ true

When $n=k+1$, $S_{k+1} < S_k$

$S_{k+1}+1 < S_k+1$

$\frac{1}{3}(S_{k+1}+1) < \frac{1}{3}(S_k+1)$

$S_{k+2} \le S_{k+1}$

$\therefore S_{n+1} < S_n$ decreasing

(d) S_n is decreasing and bounded, then it is convergent.

$\therefore$ $\lim S_n$ is exist.

$\lim S_{n+1} = \frac{1}{3}(S_n+1)$

$\because$ S_n and S_{n+1} are convergent

$\therefore$ $\lim S_n = \lim S_{n+1} = a$

$$a = \frac{1}{3}\lim(S_n+1)$$

$$3a = a+1$$

$$a = \frac{1}{2}$$

10.11 Let $t_1 = 1$ and $t_{n+1} = [1 - \frac{1}{4n^2}] \cdot t_n$ for $n \geq 1$

(a) Show $\lim t_n$ exists.

Want to show t_n is bounded

When $n = 1$ $t_1 = 1$ true

When $n = k+1$ $p(k+1)$

$$t_{k+1} = (1 - \frac{1}{4(k+1)^2}) t_{k+1} > 0$$

$\because t_k > 0$

$\therefore t_{k+1} > 0$

$$t_{n+2} = (1 - \frac{1}{4(n+1)^2}) t_{n+1} > 0$$

$t_{n+2} > 0$

$0 < t_{n+1} < t_n \leq 1 \quad \forall n$

Therefore t_n is bounded and decreasing sequence., it is exist.

(b) $\lim t_n$ is $\frac{1}{e}$

Let a_n, b_n, c_n be three sequences, such that $a_n \leqslant b_n \leqslant c_n$ and $L = \lim a_n = \lim c_n$

Show that $\lim b_n = L$

$\forall \varepsilon > 0, \exists N_1 \in \mathbb{N}$ s.t $\forall n > N$ $|a_n - L| < \frac{\varepsilon}{3}$

$\forall \varepsilon > 0, \exists N_2 \in \mathbb{N}$ s.t $\forall n > N$ $|c_n - L| < \frac{\varepsilon}{3}$

take $N = \max\{N_1, N_2\}$

$b_n - a_n \leq c_n - b_n \leq c_n - a_n$

$\leq |c_n - L - a_n + L|$

$\leq |c_n - L| + |a_n - L|$

$\leq \frac{2\varepsilon}{3}$

$|b_n - L| = |b_n - a_n| + |a_n - L| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} \sim \varepsilon$

Thus $\lim b_n = L$