Let (S_n) be a Sequance Such that

$$|S_{n+1} - S_n| < 2^{-n} \text{ for all } n \in \mathbb{N}$$

(a) WTS (S_n) is Cauchy Sequence

$\forall \varepsilon > 0, \exists N > 0$, Such that $\forall n, m > N$

$$|a_n - a_m| < \varepsilon$$

$$|a_n - a_m| \leq |a_n - a_{n-1}| + |a_{n-1} + a_{n-2}| + \cdots + |a_{m+1} - a_m|$$

$$\leq 2^{-n} + 2^{-(n+1)} + 2^{-(n+2)} + \cdots + 2^{-(m-1)}$$

$$= \frac{1}{2^n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-1}}\right)$$

$$\because 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

$$= \frac{1}{2^n} \cdot 2 \cdot \left[1 - \frac{1}{2^{m-n}}\right]$$

$$= \frac{1}{2^{n-1}} - \frac{1}{2^{m-1}}$$

$$< \frac{1}{2^{n-1}}$$

$\forall \varepsilon > 0 \quad \exists N > 0$, s.t. $\left|\frac{1}{2^{n-1}}\right| < \varepsilon \quad \forall n > N$

$\forall \varepsilon > 0 \quad \exists n, m \geq N$

$$|S_n - S_m| < \frac{1}{2^{n-1}} < \varepsilon$$

$$|S_n - S_n| < \varepsilon. \quad \forall n, m \geq N$$

Therefore S_n is convergent.

(b) No.

Let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$$|S_{n+1} - S_n| = \frac{1}{n+1} < \frac{1}{n}$$

$$\lim \frac{1}{n} = \lim \left|\frac{1}{n+1} \cdot \frac{n}{1}\right| = \lim \left|\frac{n}{n+1}\right| = \lim \left|\frac{1}{1 + \frac{1}{n}}\right| = 1$$

Therefore, S_n is divergence. Which is contradiction.

11.2 (a) $a_n = (-1)^n$

$a_n = -1, 1, -1, 1, -1, 1, \cdots$

We know the Sequence equal Constant is monotone decreasing & increasing Sequence.

Pick $1, 1, 1, 1, 1, 1$ or pick $-1, -1, -1, -1, -1$

$(a_{n_k})_{k \in \mathbb{N}}$

Therefore, a monotone subseq is (a_{2n}) or (a_{2n-1})

$b_n = \frac{1}{n}$

$b_n = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ decreasing monotone

$b_{n+1} \leq b_n \quad \forall n$

b_n is a monotone subsequence

$C_n = n^2$

$C_n = 1, 4, 9, 16, 25 \cdots$ increasing monotone.

$C_{n+1} \geq C_n$

C_n is a monotone subsequence

$d_n = \frac{6n+4}{7n-3}$

$d_n = \frac{10}{4}, \frac{16}{11}, \frac{22}{18}, \frac{28}{25}, \cdots$

$d_{n+1} \leq b_n$ decreasing monotone

d_n is a monotone subsequence.

(b) $a_n = (-1)^n$

Converge to 1 infinitely times

Converge to -1 infinitely times.

$\lim(a_n) = \lim(-1)^n = 1$ or -1

$b_n = \frac{1}{n}$

$\lim(b_n) = \lim(\frac{1}{n}) = 0$

Sequence have limit S, then all Subsequences have limit S.

Then, the set of subsequential limit is $\{0\}$.

$C_n = n^2$

$\lim C_n = \lim (n)^2 = +\infty$ diverge to $+\infty$

The set Subsequential Limit is $\{+\infty\}$

$d_n = \dfrac{6n+4}{7n-3}$

$\lim d_n = \lim \dfrac{6n+4}{7n-3} = \lim \dfrac{6+\frac{4}{n}}{7-\frac{3}{n}} = \dfrac{6}{7}$

The set of Subsequenctial limit is $\{\frac{6}{7}\}$

(c) $a_n = (-1)^n$

$\lim\sup(a_n) = 1 = \sup S$

$\lim\inf(a_n) = -1 = \inf S.$

$b_n = \frac{1}{n}$

$\lim\sup b_n = 0$

$\lim\inf b_n = 0$

$C_n = n^2$

$\lim\sup C_n = +\infty$

$\lim\inf C_n = +\infty$

$d_n = \dfrac{6n+4}{7n-3}$

$\lim\sup d_n = \dfrac{6}{7}$

$\lim\inf d_n = \dfrac{6}{7}$

(d) $a_n = (-1)^n$

a_n is not Converge, because each sequence only Can Converge to One Limit.

$b_n = \dfrac{1}{n}$

b_n is Converge. Limit exist and converge 0 infinitely times.

$C_n = n^2$

C_n is diverge to $+\infty$. Limit exist, Value is $+\infty$

$d_n = \dfrac{6n+4}{7n-3}$

d_n is converge, Limit exist and converge $\frac{6}{7}$ infinitely times.

(e) $a_n = (-1)^n$ is bounded $[-1, 1]$

$b_n = (\frac{1}{n})$ is bounded $[0, 1]$

$C_n = n^2$ is ubounded $[1, +\infty)$

$d_n = \dfrac{6n+4}{7n-3}$ is bounded $[\frac{6}{7}, \frac{5}{2}]$

11.3

(a) $S_n = \cos(\frac{n\pi}{3})$

$S_n = \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \cdots$

We can pick $-1, -1, -1, \cdots$

When $n_k = 3k$ $= \cos\frac{3k\pi}{3}$

S_{3n} is a monotone sequence.

$t_n = \frac{3}{4n+1}$

$t_n = \frac{3}{5}, \frac{3}{9}, \frac{9}{13}, \cdots$ decreasing

$b_{n+1} \leq b_n$

t_n is a monotone sequence.

$U_n = (-\frac{1}{2})^n$

$U_n = -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \cdots$

We can pick n = even, $2n$ $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \cdots$ decreasing

S_{2n} is a monotone sequence.

$V_n = (-1)^n + \frac{1}{n}$

$V_n = 0, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}$

We can pick n = even, $2n$

$\frac{1}{2}, \frac{1}{4}, \frac{1}{6} \cdots$ decreasing

V_{2n} is a monotone sequence.

(b) $S_n = \cos(\frac{n\pi}{3})$

The sequence take value $-1, -\frac{1}{2}, \frac{1}{2}, 1$ infinitely times.

So all subsequence converge to $-1, -\frac{1}{2}, \frac{1}{2}, 1$

The set of subsequential limit is $\{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$

$t_n = \frac{3}{4n+1}$

$\lim t_n = \lim \frac{3}{4n+1} = \lim \frac{\frac{3}{n}}{4+\frac{1}{n}} = 0$

$\because$ Sequence limit is 0 $\therefore$ Subsequence limit have 0.

The set of subsequential limit is $\{0\}$

$U_n = (-\frac{1}{2})^n$

$\lim U_n = \lim (\frac{(-1)^n}{(2)^n}) = 0$

$\because$ Sequence limit is 0 $\therefore$ Subsequence limit have 0.

The set of subsequential limit is $\{0\}$

$V_n = (-1)^n + \frac{1}{n}$

$\lim V_n = \lim((-1)^n + \frac{1}{n}) = +1$ or -1

$\because$ Sequence limit is ± 1 $\therefore$ Subsequence limit have ± 1

The set of subsequential limit is $\{-1, 1\}$

(C) $S_n = \cos(\frac{n\pi}{3})$

$\lim\sup S_n = 1$

$\lim\inf S_n = -1$

$t_n = \frac{3}{4n+1}$

$\lim\sup t_n = 0$

$\lim\inf t_n = 0$

$U_n = (-\frac{1}{2})^n$ $\quad$ $V_n = (-1)^n + \frac{1}{n}$

$\lim\sup U_n = 0$ $\quad$ $\lim\sup V_n = 1$

$\lim\inf U_n = 0$ $\quad$ $\lim\inf V_n = -1$

(d) $S_n = \cos(\frac{\pi n}{3})$

S_n is not converge, because each sequence only can converge to one limit.

$t_n = \frac{3}{4n+1}$

t_n is converge. it is take value 0 infinitely times.

$U_n = (-\frac{1}{2})^n$

U_n is converge, it is take value 0 infinitely times.

$V_n = (-1)^n + \frac{1}{n}$

V_n is not converge, because each sequence only can converge to one limit.

(e) $S_n = \cos(\frac{n\pi}{3})$ is bounded $[-1, 1]$

$t_n = \frac{3}{4n+1}$ is bounded $[0, \frac{3}{5}]$

$U_n = (-\frac{1}{2})^n$ is bounded $[-\frac{1}{2}, \frac{1}{4}]$

$V_n = (-1)^n + \frac{1}{n}$ is bounded $[-1, 1]$

Set of Subseq limit $= \{S\}$

11.5 Let (q_n) be an enumeration of all the rationals in the interval $(0,1]$

(a) Let S denote set of subsequential limits for (q_n)

$S \subset (0,-1]$

A sequence in interval $(0,1]$, then subsequential limit have 0

That mean subsequence converge to 0 and 1

Therefore, the set of subsequential limit is $[0,1]$

(b) $S = [0,1]$

$\lim\sup q_n = 1$

$\lim\inf q_n = 0$

2. (S_n) be a sequence in $\mathbb{R}$. We define

$\lim\sup S_n = \lim_{N\to\infty} \sup \{ S_n : n > N \}$

Lim sup is largest subsequential limits of S_n

Sup is smallest upper bound $\approx$ max

I think $\sup E = \lim\sup E$ is that seens to be correct, but is actually wrong.