1.10: Prove $(2n+1)+(2n+3)+(2n+5)+\cdots(4n-1)=3n^2$ for all positive integers n.

$P(n)=(2n+1)+(2n+3)+(2n+5)+\cdots(4n-1)=3n^2$

$n\geq 1$

When $n=1$ $\quad P(1)=2+1=3$ $\quad$ True

When $n=2$ $\quad P(2)=(4+1)+(4+3)=5+7=12=3(4)=12$ $\quad$ True

For the induction step: Suppose $P(k)$ is true

We suppose $P(k)=(2k+1)+(2k+3)+(2k+5)+\cdots(4k-1)=3k^2$

Want to show $P(k+1)$ from this

$(2(k+1)+1)+(2(k+1)+3)+(2(k+1)+5)+\cdots+(4(k+1)-1)$

$(2(k+1)+1)+(2(k+1)+3)+(2(k+1)+5)+\cdots+[(2(k+1)-1)+(2(k+1))]$

$(2(k+1)+(2(k+3)+2(k+5)+\cdots+(2k+(2k-1))+[(2(k+1)-1)+(2(k+1))]$

$+(2+\cdots+2)$

$=3k^2+[(2(k+1)-1)+(2(k+1))]+2k$

$=2k^2+(2k+1+2k+2)+2k$

$=3k^2+6k+3$

$=3(k^2+2k+1)$

$=3(k+1)^2$ $\quad$ True

Therefore $(2n+1)+(2n+3)+(2n+5)+\cdots(4n-1)=3n^2$ for all positive integers n.

1.12

(a) $n=1$ $(a+b)^1 = a+b$

$\binom{1}{0}a + \binom{1}{1}b$

$n=2$ $(a+b)^2 = a^2 + 2ab + b^2$

$= \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2$

$n=3$ $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$= \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$

(b) $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

$= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$

$= \frac{n!}{k\cdot(k-1)!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)(n-k)!}$

$= \left(\frac{n!}{k} + \frac{n!}{(n-k+1)}\right) \times \frac{1}{(k-1)!(n-k)!}$

$= \frac{n+1}{k(n-k+1)} \times \frac{n!}{(k-1)!(n-k)!}$

$= \frac{n!(n+1)}{k(n-k+1)(k-1)!(n-k)!}$

$= \frac{n!(n+1)}{k(n-k+1)!(k-1)!}$

$= \frac{n!(n+1)}{k!(n-k+1)!}$

$= \binom{n+1}{k}$

(C) $P(n) = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$

When $n=1$ $\quad (a+b)^1 = a+b$ $\quad$ true

Suppose $P(k+1)$

$$P(k+1) = (a+b)^{k+1}$$

$$= (a+b)\cdot(a+b)^k$$

$$= \left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1} + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right](a+b)$$

$$= \binom{k}{0}a^{k+1} + \binom{k}{1}a^kb + \binom{k}{2}a^{k-1}b^2 + \cdots + \binom{k}{k-1}a^2b^{k-1} + \binom{k}{k}ab^k + \binom{k}{0}a^kb + \binom{k}{1}a^{k-1}b^2 + \binom{k}{2}a^{k-2}b^3 + \cdots + \binom{k}{k-1}ab^k + \binom{k}{k}b^{k+1}$$

$$= \binom{k}{0}a^{k+1} + \left[\binom{k}{1}+\binom{k}{0}\right]a^kb + \left[\binom{k}{2}+\binom{k}{1}\right]a^{k-1}b^2 + \cdots + \left[\binom{k}{k}+\binom{k}{k-1}\right]ab^k + \binom{k}{k}b^{k+1}$$

$$= \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^kb + \binom{k+1}{2}a^{k-1}b^2 + \cdots + \binom{k+1}{k}ab^k + \binom{k+1}{k+1}b^{k+1}$$

$$\binom{k}{0} = \binom{k+1}{0} = 1 = \binom{k}{k} = \binom{k+1}{k+1}$$

$$= a^{k+1} + (k+1)a^kb + \frac{1}{2}k(k+1)a^{k-1}b^2 + \cdots + (k+1)ab^k + b^{k+1}$$

This is true for $P(k+1)$

Therefore, $P(n) = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$

2.1 Show $\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{24}$ and $\sqrt{31}$ are not rational number.

$\sqrt{3}$ The only possible rational solutions of $x^2+3=0$ are $\pm 1, \pm 3$ and none of these numbers are solution.

$\sqrt{5}$ The only possible rational solutions of $x^2+5=0$ are $\pm 1, \pm 5$ and none of these numbers are solution.

$\sqrt{7}$ The only possible rational solutions of $x^2+7=0$ are $\pm 1, \pm 7$ and none of these numbers are solution.

$\sqrt{24}$ The only possible rational solutions of $x^2+24=0$ are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$ none of these 16 numbers satisfies the equation $x^2+24=0$

$\sqrt{31}$ The only possible rational solutions of $x^2+31=0$ are $\pm 1, \pm 31$ and none of these numbers are solution.

2.2 $\sqrt[3]{2}$. Rational solution of $X^3-2=0$ are $\pm 1, \pm 2$. None of these four numbers satisfies the equation $X^3-6=0$

$\sqrt[7]{5}$. Rational solution of $X^7-5=0$ are $\pm 1, \pm 5$. None of these four numbers satisfies the equation $X^7-5=0$

$\sqrt[4]{13}$. Rational solution of $X^4-13=0$ are $\pm 1, \pm 13$. None of these four numbers satisfies the equation $X^4-13=0$

2.7

(a) $\sqrt{4+2\sqrt{3}} - \sqrt{3}$

$= \sqrt{1+3+2\sqrt{3}} - \sqrt{3}$

$= \sqrt{(1+\sqrt{3})^2} - \sqrt{3}$

$= 1+\sqrt{3}-\sqrt{3} = 1$

Therefore $\sqrt{4+2\sqrt{3}} - \sqrt{3}$ is rational number.

(b) $\sqrt{6+4\sqrt{2}} - \sqrt{2}$

$= \sqrt{2+4+4\sqrt{2}} - \sqrt{2}$

$= \sqrt{(2+\sqrt{2})^2} - \sqrt{2}$

$= 2+\sqrt{2}-\sqrt{2}$

$= 2$

Therefore $\sqrt{6+4\sqrt{2}} - \sqrt{2}$ is rational number.

3.6 (a) prove $|a+b+c| \leq |a|+|b|+|c|$ for all $a, b, c \in \mathbb{R}$

triangle inequality : $|x+y| \leq |x|+|y|$

Let $a = x$ and $b+c = y$

$$|x+y| = |a+b+c| \leq |x|+|y|$$

$\leq |a|+|b+c|$ by triangle inequality

$$\leq |a|+|b|+|c|$$

(b) prove $|a_1+a_2+\cdots+a_n| \leq |a_1|+|a_2|+\cdots+|a_n|$ for n number $a_1, a_2, \cdots, a_n$

$P(n) = |a_1+a_2+\cdots+a_n| \leq |a_1|+|a_2|+\cdots+|a_n|$

when $n=1$ $|a_1| \leq |a_1|$ true

when $n=2$ $|a_1+a_2| \leq |a_1|+|a_2|$ true

Want to show $P(k+1)$

$$P(k+1) = |a_1+a_2+\cdots+a_{k+1}|$$

$$\leq |a_1+a_2+\cdots+a_k|+|a_{k+1}|$$

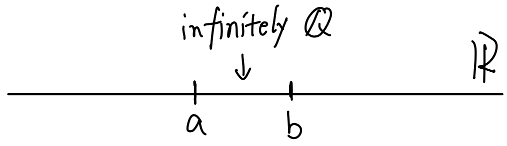
$$\leq |a_1|+|a_2|+\cdots+|a_k|+|a_{k+1}|$$

Thus, $P(k+1)$ is true.

Therefore $|a_1+a_2+\cdots+a_n| \leq |a_1|+|a_2|+\cdots+|a_n|$ for n number $a_1, a_2, \cdots, a_n$

4.11 Consider $a, b \in \mathbb{R}$ where $a < b$. Show there are infinitely many rationals between a and b.

Denseness of $\mathbb{Q}$: If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational $r \in \mathbb{Q}$ Such that $a < r < b$



Since $a < b$, then $b - a > 0$.

Assume there are finitely many rational between a and b.

$M = \{ c_1, c_2 \cdots c_n \}$ finite set, $C \in \mathbb{N}$

C_n is max number in Set M Such that $a < C_n < b$

By Denseness of $\mathbb{Q}$, There exist r in (a, b), $r \in \mathbb{Q}$ Such that

$$a < C_n < r < b$$

Which is contradiction for C_n is max.

4.14. prove $\sup(A+B) = \sup A + \sup B$

Since $a \in A$, $b \in B$, then $a \leq \sup A$, $b \leq \sup B$ and $a+b \in A+B$

Want to show $\sup(A+B) \leq \sup A + \sup B$

There exist $N \in A+B$ such that $N = a+b$

$$N = a+b \leq \sup A + \sup B$$

$\sup A + \sup B$ is upper bound of $(A+B)$

Thus, $\sup(A+B) \leq \sup A + \sup B$

Want to show $\sup(A+B) \geq \sup A + \sup B$

Since $a \in A$, $b \in B$, then $a \leq \sup A$, $b \leq \sup B$ and $a+b \in A+B$

$$\sup(A+B) \geq a+b$$

$$a \leq \sup(A+B) - b$$

$\sup(A+B)$ is upper bound of $(A+B)$ and $(a+b)$

$\sup(A+B) - b$ is upper bound of A

$\sup(A+B) - a$ is upper bound of B

We got $\sup A \leq \sup(A+B) - b$ and $\sup B \leq \sup(A+B) - a$

Since $\sup A \leq \sup(A+B) - b$, then $b \leq \sup(A+B) - \sup A$ is upper bound of B.

We got $\sup B \leq \sup(A+B) - \sup A$

Therefore $\sup B + \sup A \leq \sup(A+B)$

$\because \sup B + \sup A \leq \sup(A+B)$

$\sup(A+B) \leq \sup A + \sup B$

Therefore $\sup(A+B) = \sup A + \sup B$

4.14 (b) prove $\inf(A+B) = \inf A + \inf B$

Want to show: $\inf(A+B) \geq \inf A + \inf B$

We know $a \in A$, $b \in B$, $a+b \in A+B$

There exist $X \in (A+B)$. $X = a+b$ Such that $X \leq \inf A + \inf B$

$X = a+b \geq \inf A + \inf B$

$\inf A + \inf B$ is lower bound of $(A+B)$

Therefore $\inf(A+B) \geq \inf A + \inf B$

Want to show: $\inf(A+B) \leq \inf A + \inf B$

We know $a \in A$, $b \in B$, $a+b \in A+B$

$a \geq \inf A$ and $b \geq \inf B$ and then $a+b \geq \inf A + \inf B$

Since $a \geq \inf(A+B) - b$, then $\inf(A+B) - b$ is lower bound of A

$\inf A \geq \inf(A+B) - b$

Since $b \geq \inf(A+B) - a$, then $\inf(A+B) - a$ is lower bound of A.

$\inf B \geq \inf(A+B) - \inf A$

Hence $\inf B + \inf A \geq \inf(A+B)$

$\because \inf B + \inf A \geq \inf(A+B)$

$\inf(A+B) \geq \inf A + \inf B$

Therefore, $\inf(A+B) = \inf A + \inf B$

7.5

(a) $\lim S_n$ where $S_n = \sqrt{n^2+1} - n$

$$\lim(\sqrt{n^2+1} - n)$$

$$= \lim \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} \cdot \sqrt{n^2+1} - n$$

$$= \lim \frac{(n^2+1-n^2)\cdot}{\sqrt{n^2+1}+n}$$

$$= \lim \frac{1}{\sqrt{n^2+1}+n}$$

$$= \frac{1}{\infty} = 0$$

(b) $\lim(\sqrt{n^2+n} - n)$

$$= \lim(\sqrt{n^2+n} - n \cdot \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n})$$

$$= \lim \frac{n^2+n-n^2}{\sqrt{n^2+n}+n}$$

$$= \lim \frac{n}{\sqrt{n^2+n}+n}$$

$$= \lim \frac{1}{\frac{\sqrt{n^2+n}}{n}+1} = \lim \frac{1}{\sqrt{\frac{n^2+n}{n^2}}+1} = \lim \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$

(c) $\lim \left(\sqrt{4n^2+n} - 2n \right)$

$$= \lim \left(\sqrt{4n^2+n} - 2n \cdot \frac{\sqrt{4n^2+n} + 2n}{\sqrt{4n^2+n} + 2n} \right)$$

$$= \lim \left(\frac{n}{\sqrt{4n^2+n} + 2n} \right)$$

$$= \lim \left(\frac{1}{\frac{\sqrt{4n^2+n}}{n} + 2} \right)$$

$$= \lim \left(\frac{1}{\sqrt{\frac{4n^2+n}{n^2}} + 2} \right)$$

$$= \lim \left(\frac{1}{\sqrt{4+\frac{1}{n}} + 2} \right)$$

$$= \frac{1}{\sqrt{4+0} + 2} = \frac{1}{4}$$