13.3 (a) show d is a metric for B

For $d(X,X)=0$, $X \in S$ $\quad d(X,Y)>0$

$d(X,Y) = \sup\{ |X_j - Y_j| \mid j=1,2,\dots\}$

$d(X,X) = \sup\{ |X_j - X_j| \mid j=1,2,3\dots\} = \sup\{0\} = 0$

For $d(X,Y) = d(Y,X)$ $\quad \forall X,Y \in S.$

$d(X,Y) = \sup\{ |X_j - X_j| \mid j=1,2,\dots\}$

$d(X,Y) = \sup\{ |X_j - Y_j| \mid j=1,2\dots\} = d(Y,X)$

For $d(X,Z) \leq d(X,Y) + d(Y,Z)$

$d(X,Y) = \sup\{ |X_j - Y_j| \mid j=1,2\dots\}$

$d(X,Z) = \sup\{ |X_j - Z_j| \mid j=1,2,\dots\} = \sup\{ |X_j - Y_j + Y_j - Z_j| \mid j=1,2,\dots\}$

$\leq \sup\{ |X_j - Y_j| \mid j=1,2\} + \sup\{ |Y_j - Z_j| \mid j=1,2,\dots\}$

$= d(X,Y) + d(Y,Z)$

Therefore d is a metric for B

(b)

Let $X = (1,1,\dots)$ and $Y = (0,0,0\dots)$

$d(X,Y) = \sum_{j=1}^{\infty} |X_j - Y_j| = \sum_{j=1}^{\infty} 1 = \infty$

Therefore $d(X,Y) = \sum_{j=1}^{\infty} |X_j - Y_j|$ is not metric for B.

13.5

(a) $\bigcap\{S\setminus U: U\in \mathcal{U}\} = S\setminus \bigcup\{U: U\in \mathcal{U}\}$

$s\in\bigcap\{S\setminus U: U\in A\}$, Then $s\in S\setminus U \quad \forall U\in A$

$s\notin U \quad \forall U\in A$ and $s\notin \bigcup\{U: U\in A\}$

$\therefore s\in S\setminus\{U: U\in A\} \qquad \bigcap\{S\setminus U: U\in A\}\subseteq S\setminus\{U: U\in A\}$

$s\in S\setminus\bigcup\{U: U\in A\}$ then $s\notin\bigcup\{U: U\in A\}$

$\therefore s\notin U \quad \forall U\in A$

$s\in S\setminus U \quad \forall U\in A$

$s\in\bigcap\{S\setminus U: U\in A\}$

$S\setminus\bigcup\{U: U\in A\}\subseteq\bigcap\{S\setminus U: U\in A\}$

Therefore $\bigcap\{S\setminus U: U\in A\} = S\setminus\bigcup\{U: U\in A\}$

(b) Since $\mathcal{U}\setminus u$ is open

$\mathcal{U}: \bigcap_{u\in I} u = \bigcup_{u\in I}(\mathcal{U}\setminus u)$ is open

$\therefore \mathcal{U}\setminus\bigcap_{u\in I} u$ is close

13.7

Let $\mathcal{L}$ be an open Subset of $\mathbb{R}$. $\forall x \in \mathcal{L}$

There is $y>x$ such that $(x,y) \subset \mathcal{L}$, $\exists\ z<x$ s.t $(z,x) \subset \mathcal{L}$

Let $b = \sup\{y : (x,y) \subset \mathcal{L}\}$

Let $a = \inf\{z : (z,x) \subset \mathcal{L}\}$

$\therefore$ $a<x<b$ and $x \in I_x = (a,b) \subset \mathcal{L}$

For b: $\forall \varepsilon>0$, $b-\varepsilon<x<b$ $\quad b \notin \mathcal{L}$

For a: $\forall \varepsilon>0$, $a<x<a+\varepsilon$ $\quad a \notin \mathcal{L}$

The collection of open intervals $\{I_x\}$. $x \in \mathcal{L}$

Since each x in $\mathcal{L}$ is in I_x and each I_x is contained in $\mathcal{L}$, then $\mathcal{L} = \cup I_x$

Let (a,b) and (c,d) be two open interval, Then we must have $c<b$ and $a<d$

Since $c \notin \mathcal{L}$ then $c \notin (a,b)$ and $c \le a$

Since $a \notin \mathcal{L}$ then $a \notin (c,d)$ and $a \le c$

Therefore $b=d$. and two different interval is disjoint

Therefore every open set in $\mathbb{R}$ is the disjoint union of a finite or infinite Sequence of open intervals.

4. $S_1 = \overline{S}$ $\quad S_2 = \overline{S_1}$ $\quad S_1 = S_2$

$\overline{S} \subset \overline{\overline{S}}$ $\quad$ want to show $\overline{S} = \overline{\overline{S}}$

$\overline{\overline{S}} = \{ \overline{p} \in \overline{S} \}$

$\overline{p_n} \to \overline{p}$ for some $p \in X$, if $\forall \varepsilon > 0$, $\exists N > 0$ s.t. if $n_1 \geq N$, then $d(p_n, p) < \varepsilon$

$$|\overline{p_{n_1}} - \overline{p}| < \varepsilon.$$

$\forall \varepsilon > 0$ $\quad \exists N > 0$ $\quad , |p_{n_2} - \overline{p}| < \varepsilon$ $\quad \forall n_2 \geq N$

$|p_{n_2} - \overline{p}| \leq |p_{n_2} - \overline{p_{n_1}}| + |\overline{p_{n_1}} - \overline{p}| < \varepsilon + \varepsilon = 2\varepsilon$

Therefore, taking closure again won't make it any bigger.

5. Let $U = \bigcap$ all closed sets containing S. $= \bigcap_{\alpha \in I} U_\alpha$

$S \subseteq U_\alpha \quad \forall \alpha \in I$

$\because$ U_α is closed

$\therefore \bar{S} \subseteq U_\alpha \quad \forall \alpha \in I$

$\therefore \bar{S} \subseteq \bigcap_{\alpha \in I} U_\alpha = U$

$\therefore \bar{S} \subseteq U$

$S \cup S' = \bar{S} \qquad \therefore S \subseteq \bar{S}$

$\bar{S}$ is a closed set containing S. U is the $\bigcap$ all closed set containing S.

$\therefore U \subseteq \bar{S}$.

Therefore $\bar{S}$ is the intersection of all closed sets containing S.