12.10

Prove (S_n) is bounded if and only if $\lim\sup|S_n| < +\infty$

Since $\lim\sup|S_n| < \infty$, then $\lim_{N\to\infty}(\sup\{a_n : n \geq N\})$ is finite.

Let $A = \lim\sup a_n$, then $\forall \varepsilon > 0$, $\exists N$ s.t

$$\sup\{a_n : n \geq N\} \leq L + \varepsilon$$

Take $M = \max\{|S_1|, |S_2|, \dots |S_{n-1}|, A+\varepsilon\}$

$|S_n| < M$

Therefore S_n is bounded.

12.12

Let (S_n) be a sequence of nonnegative numbers, and for each n define

$\sigma_n = \frac{1}{n}(S_1 + S_2 + \cdots + S_n)$

(a) show $\lim\inf S_n \leq \lim\inf \sigma_n \leq \lim\sup \sigma_n \leq \lim\sup S_n$

$$\sup\{\sigma_n : n > M\} \leq \frac{1}{M}(S_1 + S_2 + \cdots + S_N) + \sup\{S_n : n > N\}$$

$$\sup\{\sigma_n : n > M\} = \sup\{\tfrac{1}{n}(S_1 + \cdots + S_n) : n > M\}$$

$$= \sup\{\tfrac{1}{n}(S_1 + \cdots + S_N) + \tfrac{1}{n}(S_{N+1} + \cdots + S_n) : n > M\}$$

$$\leq \sup\{\tfrac{1}{n}(S_1 + \cdots S_N) : n > M\} + \sup\{\tfrac{1}{n}(S_{N+1} + \cdots + S_n) : n > M\}$$

$$\sup\{X_n + Y_n : n > N\} \leq \sup\{X_n : n > N\} + \sup\{Y_n : n > N\}$$

$$\sup\{\sigma_n : n > M\} \leq \sup\{\tfrac{1}{n}(S_1 + \cdots + S_N) : n > M\} + \sup\{\tfrac{1}{n}(S_{N+1} + \cdots + S_n) : n > M\}$$

$$\leq \sup\{\tfrac{1}{M}(S_1 + \cdots + S_N)\} + \sup\{\tfrac{1}{n}(S_{N+1} + \cdots + S_n) : n > M\}$$

$$= \frac{1}{M}(S_1 + \cdots + S_N) + \frac{n-N}{n}\sup\{S_n : n > N\}$$

$$< \frac{1}{M}(S_1 + \cdots + S_N) + \sup\{S_n : n > N\}$$

$$\lim\sup \sigma_n \leq \lim_{M\to\infty}\frac{1}{M} \cdot \lim_{M\to\infty}(S_1 + \cdots + S_N) + \lim_{M\to\infty}\sup\{S_n : n > N\}$$

$$= 0 \cdot \lim_{M\to\infty}(S_1 + \cdots + S_N) + \sup\{S_n : n > N\}$$

$$\lim_{N\to\infty} \limsup \sigma_n \le \lim_{N\to\infty} \sup\{S_n : n > N\}$$

$$\limsup \sigma_n \le \limsup S_n.$$

(b) If S_n exist

$\liminf S_n = \lim S_n = \limsup S_n$

$\liminf S_n = \liminf \sigma_n = \limsup \sigma_n = \limsup S_n$

$\because \liminf \sigma_n = \limsup \sigma_n$

$\therefore \lim \sigma_n$ exist.

$\lim \sigma_n = \lim S_n$

(C) $S_n = 1 + (-1)^n$

14.2

(a) $\sum \frac{n-1}{n^2}$

$$n-1 > \frac{n}{2}$$

Let $a_n = \frac{n-1}{n} \geq b_n = \frac{1}{2n}$

$\frac{1}{2n}$ is diverge by harmonic series

Therefore, $\sum \frac{n-1}{n^2}$ diverges by Comparison Test.

(b) $(-1)^n$ $\quad -1, 1, -1, 1, \cdots$

when n is odd $= -1$

when n is even $= 1$

Therefore, The $(-1)^n$ is diverge.

(c) $\sum \frac{3n}{n^3}$

$$a_n = \frac{3n}{n^3} = \frac{3}{n^2} = b_n$$

$$\left|\frac{3n}{n^3}\right| \leq \frac{3}{n^2}$$

The series $\sum \frac{3n}{n^3}$ Converges by Comparison Test.

(d) $\sum \frac{n^3}{3^n}$

$$\lim \left|\frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}\right|$$

$$= \lim \left|\frac{(n+1)^3}{3} \cdot \frac{1}{n^3}\right|$$

$$= \frac{1}{3} \lim \left|\left(\frac{n+1}{n}\right)^3\right| = \frac{1}{3} \lim \left|\left(1+\frac{1}{n}\right)^3\right| = \frac{1}{3} \times 1 = \frac{1}{3} < 1$$

The $\sum \frac{n^3}{3^n}$ Converges by the Ratio Test.

(e) $\sum \frac{n^2}{n!}$

$$\lim \left| \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2} \right|$$

$$= \lim \left| \frac{(n+1)^2}{(n+1)n!} \cdot \frac{n!}{n^2} \right|$$

$$= \lim \left| \frac{n+1}{n^2} \right|$$

$$= \lim \left| \frac{1}{n} + \frac{1}{n^2} \right| = 0 < 1$$

The $\sum \frac{n^2}{n!}$ Converges by the Ratio Test.

(f) $\sum \frac{1}{n^n}$

$$\lim \left| \frac{1}{n^n} \right|^{\frac{1}{n}} = \lim \left| \frac{1}{n} \right| = 0 < 1$$

$\sum \frac{1}{n^2}$ is convergent by the Root Test.

(g) $\sum \frac{n}{2^n}$

$$\lim \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right|$$

$$= \frac{1}{2} \lim \left| \frac{n+1}{n} \right|$$

$$= \frac{1}{2} \lim \left| 1 + \frac{1}{n} \right| = \frac{1}{2} \times 1 = \frac{1}{2} < 1$$

The $\sum \frac{n}{2^n}$ Converges by the Ratio Test.

14.10

Let $S_n = (2+(-1)^n)^n$.

Then $\limsup |S_n|^{\frac{1}{n}} = 3 > 1$

divegers.

$$\left| \frac{S_{n+1}}{S_n} \right| = \frac{(2+(-1)^{n+1})^{n+1}}{(2+(-1)^n)^n}$$

$\limsup \left(\frac{S_{n+1}}{S_n} \right) = \infty > 1$ odd

$\liminf \left(\frac{S_{n+1}}{S_n} \right) = 0 < 1$ even

6. (a)

$$a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{2\sqrt{n+1}}$$

$p = \frac{1}{2} < 1$ $\therefore$ a_n is diverges.

(b) $$\frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \le \frac{1}{2n\sqrt{n}} = \frac{1}{2n^{\frac{3}{2}}}$$

$$P = \frac{3}{2} > 1$$

Therefore $\sum a_n$ is convergent.

(c) $a_n = (\sqrt[n]{n} - 1)^n$

$\lim (\sqrt[n]{n} - 1) = 1 - 1 = 0$

By the root Test, $\sum a_n$ Converges

(d) $a_n = \frac{1}{1+z^n}$

$|z| \le 1$, then $a_n \ge \frac{1}{2}$ diverge.

$|z| > 1$, then $r = \frac{1}{|z|} < 1$ Converges.

7. $\sum \frac{\sqrt{a_n}}{a_n}$

By Cauchy $\quad \sum_{n=1}^{k} a_n \sum_{n=1}^{k} \frac{1}{n^2} \geq \sum_{n=1}^{k} a_n \frac{\sqrt{a_n}}{n}$

$\forall n \in \mathbb{N}$. $\sum a_n$ and $\sum \frac{1}{n^2}$ are convergent.

$\therefore \sum_{n=1}^{k} a_n \frac{\sqrt{a_n}}{n}$ is bounded. $\quad \frac{\sqrt{a_n}}{n} \geq 0 \quad \forall n$

Therefore $\sum \frac{\sqrt{a_n}}{n}$ is convergent.

9.

(a) $\quad \sum n^3 z^n$

$\alpha = \lim\sup (n^3)^{\frac{1}{n}} = 1$

$R = \frac{1}{\alpha} = \frac{1}{1} = 1$

(b) $\sum \frac{2^n}{n!} z^n$

$\lim\sup \left(\frac{2^n}{n!}\right)^{\frac{1}{n}} \qquad \frac{2}{(n!)^{\frac{1}{n}}}$

$R = +\infty$

(c) $\sum \frac{2^n}{n^2} z^n$

$\lim\sup \left(\frac{2^n}{n^2}\right)^{\frac{1}{n}} \qquad \frac{2}{(n^2)^{\frac{1}{n}}} = \frac{2}{n^{\frac{1}{n}} \cdot n^{\frac{1}{n}}} = 2.$

$\lim\sup \left(\frac{2^n}{n^2}\right)^{\frac{1}{n}} = 2$

$R = \frac{1}{\alpha} = \frac{1}{2}$

(d) $\sum \frac{n^3}{3^n} z^n$

$\lim\sup \left(\frac{n^3}{3^n}\right)^{\frac{1}{n}} \qquad \frac{(n^3)^{\frac{1}{n}}}{3} = \frac{n^{\frac{1}{n}} \cdot n^{\frac{1}{n}} \cdot n^{\frac{1}{n}}}{3}$

$R = \frac{1}{\alpha} = \frac{1}{\frac{1}{3}} = 3$

11. $a_n > 0$, $S_n = a_1 + \cdots + a_n$

(a) $\sum \dfrac{a_n}{1+a_n}$

Assume $\dfrac{a_n}{1+a_n}$ is convergent

$a_n \to 0$, $\varepsilon' = 1$ $\exists N_1 > 0$

$|a_n - 0| < 1$

$a_n < 1$

$\sum \dfrac{a_n}{1+a_n}$ $\forall \varepsilon > 0$, $\exists N_2 > 0$

$$\left| \frac{a_m}{1+a_m} + \cdots + \frac{a_n}{1+a_n} \right| < \varepsilon \qquad \text{all } n > m \geq N_2$$

Take $N = \max(N_1, N_2)$

$$\begin{aligned} \varepsilon &> \frac{a_m}{1+a_m} + \cdots + \frac{a_n}{1+a_n} \\ &> \frac{a_m}{1+1} + \cdots + \frac{a_n}{1+1} \\ &= \frac{a_m + \cdots + a_n}{2} \qquad \forall n > m \geq N \end{aligned}$$

$a_m + \cdots + a_n < 2\varepsilon$

Which is contradiction.

(b) $$\frac{a_{N+1}}{S_{N+1}}+\cdots+\frac{a_{N+k}}{S_{N+k}} \ge 1-\frac{S_N}{S_{N+k}}$$

$$\frac{a_{N+1}}{S_{N+1}}+\cdots+\frac{a_{N+k}}{S_{N+k}} \ge \frac{a_{N+1}}{S_{N+k}}+\cdots+\frac{a_{N+k}}{S_{N+k}}$$

$$= \frac{a_{N+1}+\cdots+a_{N+k}}{S_{N+k}}$$

$$= \frac{S_{N+k}-S_N}{S_{N+k}}$$

$$= 1-\frac{S_N}{S_{N+k}}$$

If $\sum \frac{a_n}{S_n}$ Converges, $\forall \varepsilon>0 \ \exists N$

$$\frac{a_m}{S_m}+\cdots+\frac{a_n}{S_n} < \varepsilon \qquad \forall n>m\ge N$$

Fix $m=N$ and $n=N+k$

$$\varepsilon > \frac{a_m}{S_m}+\cdots+\frac{a_n}{S_n}$$

$$= \frac{a_N}{S_N}+\cdots+\frac{a_{N+k}}{S_{N+k}}$$

$$\ge 1-\frac{S_N}{S_{N+k}} \qquad \forall k\in N$$

Since $\sum a_n$ diverges and $a_n>0$

Take $\varepsilon=\frac{1}{2}$ which is contradiction

$\sum \frac{a_n}{S_n}$ diverges.

(c) $S_{n-1}\le S_n$

$$\frac{1}{S_n^2} \le \frac{1}{S_nS_{n-1}}$$

$$\frac{a_n}{S_n^2} \le \frac{a_n}{S_nS_{n-1}} = \frac{S_n-S_{n-1}}{S_nS_{n-1}}$$

$$\frac{a_n}{S_n^2} \le \frac{1}{S_{n-1}}-\frac{1}{S_n}$$

$$\sum_{n=2}^{k}\frac{a_n}{S_n^2} \le \sum_{n=2}^{k}\left(\frac{1}{S_{n-1}}-\frac{1}{S_n}\right) = \frac{1}{S_1}-\frac{1}{S_n}$$

Since $\sum a_n$ diverges. $\sum \frac{a_n}{S_n^2}$ Converges.

(d) $\sum \frac{a_n}{1+na_n}$ may converge or diverge.

$\sum \frac{a_n}{1+n^2 a_n}$ converges.

put $a_n = \frac{1}{n}$ $\quad \frac{a_n}{1+na_n} = \frac{1}{2n}$

$\sum \frac{a_n}{1+na_n} = 2\sum \frac{1}{n}$ diverges.

$a_n = \frac{1}{n(\log n)p}$

$p > 1 \quad n \geq 2$

$$\frac{a_n}{1+na_n} = \frac{1}{n(\log n)^{2p}((\log n)^p+1)}$$

$$< \frac{1}{2n(\log n)^{3p}}$$

$$\sum \frac{a_n}{1+n^2 a_n} = \sum \frac{1}{\frac{1}{a_n}+n^2} < \sum \frac{1}{n^2}$$

Therefore $\sum \frac{a_n}{1+n^2 a_n}$ converges.