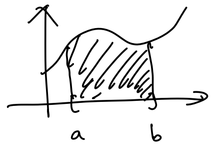
Lebesgue measure and integral.

- <u>Riemann integral</u> over $\mathbb{R}$



$$\int_a^b f(x)\,dx = \text{area under the curve}$$

At least, piecewise continuous functions are Riemann integrable.
(in particular, piecewise constant function).

shortcoming:

- the underlying space is like $\mathbb{R}$, $\mathbb{R}^n$, (the domain of integration here is only $[a,b]$).
  bounded.
- only bounded functions are considered.
- If $f_n \to f$ <u>pointwise</u>, and f_n is Riemann integrable, it is not true that f is Riemann integrable.

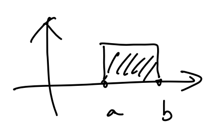
- <u>Lebesgue integral</u> : $\int_\Omega f\,\underline{dx}$. $\quad$ some subset $\Omega \subset \mathbb{R}^n$, $dx = dx_1 \cdots dx_n$.
  - what Ω do we allow? (Lebesgue measurable sets).
  - what f do we allow? (Lebesgue integrable function)

- Lebesgue measure : $m(\Omega) = \int_\Omega 1 \cdot dx$ $\quad$ constant function 1

intuitively, (1) if $\Omega \subset \mathbb{R}$,
then $m(\Omega) = \underline{\text{length}}$ of Ω

(2) $\Omega \subset \mathbb{R}^2$, $m(\Omega) = \underline{\text{area}}$ of Ω

(3) $\Omega \subset \mathbb{R}^3$: $m(\Omega) = \underline{\text{volume}}$ of Ω.



$$\left(\int_a^b 1\,dx = b-a = |[a,b]|. \right)$$

we know length of a interval, $|[a,b]| = b-a$.

Can we "consistantly" define length (or generally, measure) for any subset of $\mathbb{R}^n$? A few desirable properties

① monotone : If $A \subset B \subset \mathbb{R}^n$, then $m(A) \leq m(B)$.

Ex: $A = (0,1)$, $B = [0,1]$. in $\mathbb{R}$. $m(A) = m(B) = 1$.
$A = \{1\}$, $B = \{1,2\}$. in $\mathbb{R}$ $m(A) = m(B) = 0$.

② additivity : If $A \cap B = \phi$, then $m(A \cup B) = m(A) + m(B)$.

③ translation invariance : $\forall x \in \mathbb{R}^n$, $E \subset \mathbb{R}^n$,

$m(E) = m(x+E)$

$$\left(\begin{aligned} x+E &= \{x+a \mid a \in E\}. \\ \text{e.g. } 3 + [1,2] &= [3+1, 3+2] \\ &= [4,5] \end{aligned} \right)$$

Trouble : not possible to define such a measure on all subsets on $\mathbb{R}^n$.

- Read Tao 7.3 for an example.
- Read wiki: Banach - Tarski paradox:

a unit ball in $\mathbb{R}^3$ = disjoint union of finitely many pieces.
→ after rotation and translation, one can assemble them to 2 unit balls.

Cure : • Restrict the $\overbrace{\text{class}}^{\text{will be called "measurable"}}$ of subsets in $\mathbb{R}^n$, to which we assign a measure.

Desired Properties / Axioms of measurable subsets: (Tao - II : Page 180).

Let M_n denote the set of measurable subsets in $\mathbb{R}^n$. We want

(1) If $U \subset \mathbb{R}^n$ is open, then

· notation: if S is a set. 2^S denote the set of subsets in S.

$\mathcal{U} \in \mathcal{M}_n$. ($\mathcal{U}$ is measurable)

(2) If $\mathcal{U} \in \mathcal{M}_n$, then $\mathcal{U}^c = \mathbb{R}^n \setminus \mathcal{U}$ is also in $\mathcal{M}_n$.

(3). If $\mathcal{U}, V \in \mathcal{M}_n$, then $\mathcal{U} \cap V$ and $\mathcal{U} \cup V$ are measurable.

(2), (3) implies $\mathcal{M}_n$ is a <u>Boolean algebra</u>.

i.e. a set with 3 operations

① NOT. ② AND ③ OR

$(\ )^c$. $(\ ) \cap (\ )$, $(\ ) \cup (\ )$

(4) We want $\mathcal{M}_n$ to be a <u>countable σ-algebra</u>, namely.

If $\mathcal{U}_1, \mathcal{U}_2, \cdots$ is a sequence of measurable sets.

then $\bigcup_n \mathcal{U}_n$ is measurable, $\bigcap_n \mathcal{U}_n$ is measurable.

$$m_n : \underline{\mathcal{M}_n} \to \mathbb{R}_{\geq 0}.$$

Axioms for <u>Lebegues measure</u>. (read Tao)

<u>Thm</u>: There exists a definition of $\mathcal{M}_n$, and Lebegue measure, satisfying all the axioms.

<u>Definition</u> (<u>outer</u> measure): $\forall E \subset \mathbb{R}^n$

$$m^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mathrm{vol}(B_i), \quad \bigcup_{i=1}^{\infty} B_i \supset E, \quad B_i \subset \mathbb{R}^n \text{ are open boxes.} \right\}$$

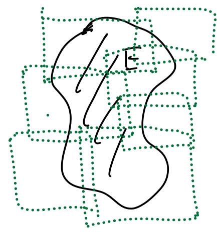
$m^*(E) \in [0, \infty]$.

open box in $\mathbb{R}^1$: (a, b)

——— in $\mathbb{R}^n$: $(a_1, b_1) \times (a_2, b_2)$

$\mathrm{vol}(B) = \prod (b_j - a_j)$

$B = \prod_{j=1}^{n} (a_j, b_j)$



- outer measure is defined for ALL subsets in $\mathbb{R}$.

<u>In discussion</u>: try prove properties of $m^*(E)$.

Lemma 7.2.5, <u>7.2.6.</u> $m^*(\text{box}) = \text{vol}(\text{box})$