1. Outer measure of any subset in $\mathbb{R}^n$.

Def: $\forall E \subset \mathbb{R}^n$

$$m^*(E) = \inf \left\{ \sum_{i=1}^{\infty} vol(B_i), \quad \{B_i\} \text{ is an } \underbrace{\text{open cover}}^{\text{countable}} \text{ of } E \text{ by boxes} \right\}$$

Lemma. 7.2.5

Lemma 7.2.5 (Properties of outer measure). *Outer measure has the following six properties:*

(v) *(Empty set) The empty set $\emptyset$ has outer measure $m^*(\emptyset) = 0$.* ← no box is needed for cover.

(vi) *(Positivity) We have $0 \le m^*(\Omega) \le +\infty$ for every measurable set Ω.*

↖ by def, inf over non-negative numbers. ∴ result is ≥ 0.

(vii) *(Monotonicity) If $A \subseteq B \subseteq \mathbf{R}^n$, then $m^*(A) \le m^*(B)$.*

(viii) *(Finite sub-additivity) If $(A_j)_{j\in J}$ are a finite collection of subsets of $\mathbf{R}^n$, then $m^*(\bigcup_{j\in J} A_j) \le \sum_{j\in J} m^*(A_j)$.*

(x) *(Countable sub-additivity) If $(A_j)_{j\in J}$ are a countable collection of subsets of $\mathbf{R}^n$, then $m^*(\bigcup_{j\in J} A_j) \le \sum_{j\in J} m^*(A_j)$.*

(xiii) *(Translation invariance) If Ω is a subset of $\mathbf{R}^n$, and $x \in \mathbf{R}^n$, then $m^*(x+\Omega) = m^*(\Omega)$.*

✓ if $\{B_i\}$ covers Ω then $\{x+B_i\}$ covers $x+\Omega$.

Pf: (vii) For any open cover $\{B_i\}$ of B, it is also an open cover of A. (And, if, $M, N \subset \mathbb{R}$, $M \supset N$. then $\inf M \le \inf N$.)

Thus, $m^*(A) \le m^*(B)$

(viii). Finite sub-additivity. $\overbrace{m^*(A\cup B) \le m^*(A) + m^*(B)}^{\text{W.T.S.}}$.

Try proving ① $m^*(A) + m^*(B) \geq$ (total Area of some covering of A, and covering of B) $- \varepsilon$

then ② (total area) $\geq m^*(A\cup B)$.

thus, $\forall \varepsilon > 0$, $m^*(A) + m^*(B) \geq m^*(A \cup B) - \varepsilon$.

$$\Rightarrow \quad m^*(A) + m^*(B) \geq m^*(A \cup B).$$

$\because$ $m^*(A) = \inf \{ \sum Vol(B_i) \mid \{B_i\} \text{ cover } A \}$

$\therefore$ $\forall \varepsilon > 0$, $\exists$ covering $\{B_i\}$, s.t. $\sum_i |B_i| \leq m^*(A) + \varepsilon$.

similarly do it for B. Then take the union of the 2 countable covers. to get a cover of $A \cup B$.

$$m^*(\bigcup_{j=1}^{\infty} A_j) \leq \sum m^*(A_j).$$

(x). W.T.S. $\forall \varepsilon > 0$, there exists a collection of open covers. $\{B_i^{(j)}\}$ for A_j. such that

$$m^*(\bigcup_{j=1}^{\infty} A_j) \leq \sum_j m^*(A_j) + \varepsilon. \qquad (\ast) \checkmark$$

$$= \sum_{j=1}^{\infty} \left(m^*(A_j) + \frac{\varepsilon}{2^j} \right)$$

We can find open cover $\{B_i^{(j)}\}$ for A_j, s.t.

$$m^*(A_j) + \frac{\varepsilon}{2^j} \geq \sum_{i=1}^{\infty} |B_i^{(j)}|$$

And $$\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} |B_i^{(j)}| \right) \geq m^*(\bigcup_{j=1}^{\infty} A_j)$$

Prop 7.2.6.

Proposition 7.2.6 (Outer measure of closed box). *For any closed box*

$$B = \prod_{i=1}^{n} [a_i, b_i] := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_i \in [a_i, b_i] \text{ for all } 1 \leq i \leq n\},$$

we have

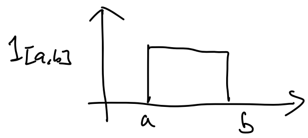
$$m^*(B) = \prod_{i=1}^{n} (b_i - a_i).$$

Recall: • compact set in $\mathbb{R}^n$ $\Longleftrightarrow$ closed and bounded.

• Riemann integral:

1-dim $\quad \text{vol}([a,b]) = b-a = \int_a^b 1\, dx = \int_{\mathbb{R}} 1_{[a,b]}\, dx$

indicator function.



$1_{[a,b]}(x) = \begin{cases} 1 & x \in [a,b] \\ 0 & \text{else.} \end{cases}$

$dx = dx_1\, dx_2 \cdots dx_n.$

n-dim. $\quad \text{Vol}(\overbrace{[a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n]}^{B}) = \int_{\mathbb{R}^n} 1_B(x)\, dx$

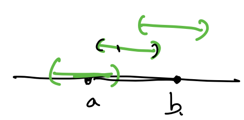
same is true for open boxes.

Pf: $\forall \varepsilon > 0$ It's clear that, we can choose a open box, slightly larger than B, to cover B, thus.

$$m^*(B) \le \text{Vol}(B) + \varepsilon \qquad \forall \varepsilon > 0 \Rightarrow m^*(B) \le \text{Vol}(B)$$

($n=1$ case). because $B = [a,b]$ is compact, hence any open cover of B can be reduced to a finite subcover. Let $\{B_i\}_{i=1}^N$ be a finite open cover of B. WTS.

$$\sum_{i=1}^N |B_i| \ge \text{Vol}(B). \quad (*)$$



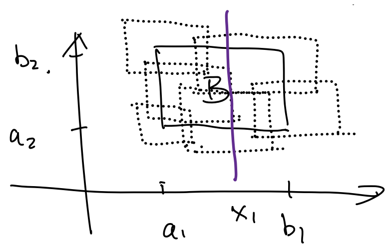
let $f_i(x) = 1_{B_i}(x)$, then

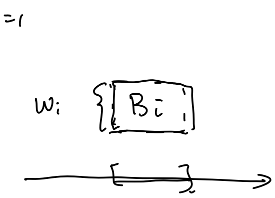
$$\sum_{i=1}^N |B_i| = \sum_{i=1}^N \left(\int_{\mathbb{R}} f_i\, dx\right) = \int_{\mathbb{R}} \underbrace{\sum_{i=1}^N f_i}_{f(x)}\, dx \ge \int 1_B\, dx = \text{Vol}(B).$$

claim: $f(x) \ge 1_B(x)$ indeed, $B \subset \bigcup_{i=1}^N B_i$

thus $1_B \le \sum 1_{B_i}$

($n=2$ case). WTS. given any finite cover $\{B_i\}_{i=1}^N$ of B. that $\sum^N |B_i| \ge |B|$





again $\quad |B_i| = \int_{\mathbb{R}^2} \underline{1_{B_i}(x_1, x_2)}\, dx_1\, dx_2 = \int_{\mathbb{R}} w_i \cdot 1_{B_{i,1}}(x_1)\, dx_1$

going to integrate along x_2

$$\sum_{i=1}^{N} \int 1_{B_i}(x)\, dx_1\, dx_2$$

$$= \int_{\mathbb{R}^2} \sum_{i=1}^{N} 1_{B_i}(x)\, dx_1\, dx_2 = \int_{\mathbb{R}} \Big(\overbrace{\int \sum_i 1_{B_i}(x_1, x_2)\, dx_2}^{f(x_1)} \Big)\, dx_1$$

Claim: $f(x_1) \geq 1_{[a_1, b_1]}(x_1) \cdot \underbrace{|b_2 - a_2|}_{\text{height of } B}$.

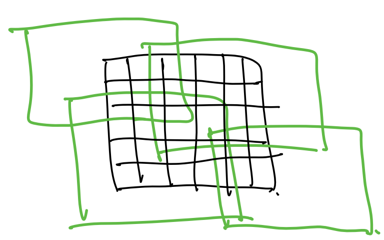
$\begin{cases} \text{if } x_1 \in [a_1, b_1], & f(x_1) \geq |b_2 - a_2| \checkmark \\ \text{if } x_2 \notin [a, b_1], & f(x_1) \geq 0 \text{ is true.} \end{cases}$

this follows by induction hypothesis (the $n=1$ case). applied to the line with the given x_1.

$$\geq \int_{\mathbb{R}} 1_{[a_1, b_1]} (b_2 - a_2)\, dx_1 = (b_2 - a_2)(b_1 - a_1)$$
$$= vol(B).$$

The general n is by induction.

Pugh:



- divide B into ~~small~~ grids of smaller boxes. so that each small box is contained in some B_i

• then $\text{Vol}(B) = \sum$ vol of grid small boxs
$\lesssim \sum$ Vol of open cover B_i

or
open, closed, half open / half closed e.g
$[a_1, b_1) \times (a_2, b_2]$

Cor: outer measure of any box
= vol (box)

Last time in discussion: (1) $m^*(\mathbb{N}) = 0$ (why? by countable sub-additivity)

$$m^*(\mathbb{N}) \leq \sum_{i=0}^{\infty} m^*(\{i\}) = \sum_{i=0}^{\infty} 0 = 0.$$

(2) similarly $m^*(\mathbb{Q}) = 0$. $\because$ $\mathbb{Q}$ is countable.

(3). $m_1^*(\mathbb{R}) = \infty$ by monotonicity

$m_n^*(E)$ is measure of $E \subset \mathbb{R}^n$.

$\because$ $m_1^*((-R, R)) = 2R$.

$\therefore$ $m_1^*(\mathbb{R}) \geqslant 2R$ $\forall R > 0$.

$\therefore$ $m_1^*(\mathbb{R}) = +\infty$.

Skip Tao. §7.3

idea: (1) construct a "weird" subset $E \subset [0, 1]$.

(2) $[-1,2] \supset \bigsqcup_{q \in [-1,1] \cap \mathbb{Q}} q + E \supset [0,1]$

then trouble: additivity would fail.

$$\because \quad m^*\Big(\bigsqcup_{q \in [-1,1] \cap \mathbb{Q}} q+E\Big) = \sum_{q \in [-1,1] \cap \mathbb{Q}} m^*(q+E) = \sum_{q \in \dots} m^*(E) = \begin{matrix} +\infty ? \\ 0 ? \end{matrix}$$

$$\underset{1}{\overset{\shortparallel}{m^*([0,1])}} \leq m^*\Big(\bigsqcup_q q+E\Big) \leq m^*([-1,2]) = 3$$