- <u>one remark</u> about last homework: we need.

  - m^*(any box) = volume of a box (about Lebesgue measure)

we know m^*(closed box) = volume

m^*(open box) = volume.

If we have any "half open half closed" box, B.

$B^\circ \subset B \subset \overline{B}$, then

$$\underbrace{vol(B^\circ) = m^*(B^\circ) \leq m^*(B) \leq m^*(\overline{B}) = vol(\overline{B})}_{\because =}$$

$\therefore$ $m^*(B) = vol(\overline{B}) = vol(B^\circ) = \prod (b_i - a_i)$

- <u>Lemma 7.4.7</u>: If A, B are measurable, $A \subset B$,

then $B \setminus A$ is measurable, and

$$m^*(B \setminus A) = m^*(B) - m^*(A).$$

<u>Pf</u>:
- $B \setminus A = B \cap A^c$ is measurable.

  ($\because$ A is meas.. $\therefore$ A^c is meas.)

  ($\because$ B and A^c is meas. $\therefore$ $B \cap A^c$ is mea..)

- <u>WTS</u> $m^*(B) = m^*(A) + m^*(B \setminus A)$.

  this follows from measurability of A, applied to test set B

  ($\because$ $m^*(B) = m^*(\underbrace{B \cap A}_{=A}) + m^*(B \cap A^c)$)

#

- <u>Prop</u> (Countable Additivity): Let $\{E_j\}_{j=1}^{\infty}$ be a countable collection of disjoint measurable set.

we want to show

- $E = \bigcup_{j=1}^{\infty} E_j$ is measurable
- $m^*(E) = \sum_{j=1}^{\infty} m^*(E_j)$

Pf: · To prove measurability, we want to show, $\forall A \subset \mathbb{R}^n$,

(*) $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$.

- Define $F_N = \bigsqcup_{j=1}^{N} E_j$. We know,
  - F_N is measurable. (finite union of meas sets)
  - $m^*(F_N) = \sum_{j=1}^{N} m^*(E_j)$

- If we replace E by F_N, $\because E \supset F_N$, $E^c \subset F_N^c$.

$\therefore$ $m^*(A \cap E) \gtrsim m^*(A \cap F_N)$ (need fixing)

$m^*(A \cap E^c) \leq m^*(A \cap F_N^c)$

- To prove (*), we need "$\leq$" (easy) and "$\gtrsim$"

finite subadditivity

$$m^*(A \cap E) \leq \sum_{j=1}^{\infty} m^*(A \cap E_j) \quad \text{by countable sub-additivity}$$

$$= \sup_N \left(\sum_{j=1}^{N} m^*(A \cap E_j) \right)$$

$$= \sup_N m^*(A \cap F_N) \quad \left(\text{by finite additivity}\right)$$

Thus: $m^*(A \cap E) + m^*(A \cap E^c) \leq \left[\sup_N m^*(A \cap F_N)\right] + m^*(A \cap E^c)$

$\leq \sup_N \left(m^*(A \cap F_N) + m^*(A \cap E^c) \right)$

$$\le \sup_N \left(m^*(A \cap F_N) + m^*(A \cap F_N^c) \right)$$
$$= \sup_N \left[m^*(A) \right] = m^*(A).$$

- $m^*(E) \le \sum_j m^*(E_j)$ by sub-additivity.

$$m^*(E) \underset{\uparrow\text{ by monotonicity}}{\ge} m^*(F_N) = \sum_{j=1}^N m^*(E_j)$$

sup over N, we have $m^*(E) \ge \sum_{j=1}^\infty m^*(E_j)$

$$\therefore\ m^*(E) = \sum_{j=1}^\infty m^*(E_j)$$ #.

<u>Prop. 7.4.9</u>: The set of measurable sets forms a <u>σ-algebra.</u> i.e. given any countable collection Ω_j of measurable sets, $\bigcap_{j=1}^\infty \Omega_j$ and $\bigcup_{j=1}^\infty \Omega_j$ are measurable.

<u>Pf</u>: • Let's consider $\Omega = \bigcup_{j=1}^\infty \Omega_j$. Let

$\Omega_0 = \phi$. $\Omega_N = \bigcup_{j=1}^N \Omega_j$, let $E_N = \Omega_N \setminus \Omega_{N-1}$.

then. $\{\Omega_N\}$ are measurable, $\{E_N\}$ are measurable.

Since $\Omega = \bigcup_{j=1}^\infty E_j$ $\therefore$ Ω is measurable

- $\bigcap_{j=1}^\infty \Omega_j = \left(\bigcup_{j=1}^\infty \Omega_j^c \right)^c$ $\because$ complement & countable union preserve measurability

$\therefore$ this is measurable.

Lemma 7.4.10 : All open sets in $\mathbb{R}^n$ can be written as a countable union of open boxes.

Recall some topology:

- topology for a metric space (X, d).
  - open ball $B(x,r) = \{ y \in X \mid d(y,x) < r \}$, $x \in X$, $r > 0$ real.
  - open sets in X are generated from open balls, by taking finite intersection and arbitrary union.
  - equivalently, $U \subset X$ is open, iff $\forall x \in U$, $\exists r > 0$, s.t. $B(x,r) \subset U$.

- topology for product space :

  If X, Y are top. spaces, then $X \times Y$ can be endowed with product topology., i.e. $W \subset X \times Y$ is open, if $\forall (x,y) \in W$, $\exists U \subset X$, $V \subset Y$ open. s.t. $(x,y) \in U \times V \subset W$.

- topology on $\mathbb{R}^2$:
  - can be generated by open balls. (using Euclidean metric on $\mathbb{R}^2$)
  - can be generated by open box.

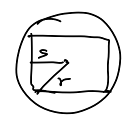
Pf: · Consider the set of "rational boxes". A box $\prod_{i=1}^{n} (a_i, b_i)$ is rational, if $a_1, b_1, \cdots, a_n, b_n \in \mathbb{Q}$.

- $\{$The collection of rational open boxes$\} \subset \mathbb{Q}^{2n}$.
  is countable.
  ($\because$ $\mathbb{Q}$ is countable, finite product of countable set is countable
  and subset of countable set is countable.)

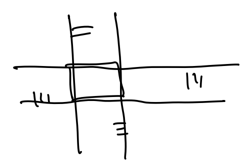
- Suffice to show that, every open set in $\mathbb{R}^n$ is a union of rational boxes. i.e. If U is open, $x \in U$, we want to find a rational open box B, s.t. $x \in B \subset U$.

  $\because$ U is open $\therefore$ $\exists r>0$. s.t. $x \in B(x, r) \subset U$.

  claim: $\exists$ rational box B. s.t.
  $$x \in B \subset B(x, r).$$



Prop: Open sets in $\mathbb{R}^n$ are measurable.



Pf:
- open boxes are measurable.
- a open set is a <u>countable</u> union of open boxes.

Alternative definition of measurable set.

<u>Def 2</u>: A subset $E \subset \mathbb{R}^n$ is measurable, if $\forall \varepsilon > 0$, there exist an open set U, s.t. $U \supset E$.
$$m^*(U \setminus E) < \varepsilon.$$

in discussion: prove that all the properties of measurable sets (Lemma in 7.4) can be derived using this <u>Def 2</u>.