17) a) Fix two real numbers a, b $0 < a < b$.
Define $f = (f_1, f_2, f_3)$, $f : \mathbb{R}^2 \to \mathbb{R}^3$, by

$$f_1(s,t) = (b + a\cos s)\cos t$$
$$f_2(s,t) = (b + a\cos s)\sin t$$
$$f_3(s,t) = a\sin s$$

$$\nabla f_1 = \begin{bmatrix} -a\sin s\cos t \\ -(b + a\cos s)\sin t \end{bmatrix} = 0$$

$$\Rightarrow \quad a\sin s\cos t = (b + a\cos s)\sin t = 0.$$

$$\cos s = 0 \Rightarrow \sin s \neq 0.$$
$$\cos t = 0 \Rightarrow \sin t \neq 0$$

so if $\nabla f_1 = 0$, since $a < b$ and $\cos s < 1$, so $a\cos s < b$, we must have $\sin t = \sin s = 0$
so $(t, s) \in \{(0,0), (0,\pi), (\pi, 0), (\pi, \pi)\}$
Then

$$f_1(s,t) = f(0,0)$$
$$\text{or } f(0,\pi)$$
$$\text{or } f(\pi, 0)$$
$$\text{or } f(\pi,\pi)$$

$$\Rightarrow \quad f(s,t) = \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix}$$

where $v_1 = (b + a)$
or $v_1 = (b - a)$
or $v_1 = (-b + a)$
or $v_1 = (-b - a)$.

This is the typical parametrization of a torus.

b) If $\nabla f_2(f^{-1}(q)) = 0$ for $q \in K$, then

$$a\cos s = 0 \quad \text{since} \quad \nabla f_2 = \begin{bmatrix} a\cos s \\ 0 \end{bmatrix}$$

so

$$s = \pi/2 \quad \text{or} \quad s = 3\pi/2$$

Then $f_2 = \begin{bmatrix} b\cos t \\ b\sin t \\ a \end{bmatrix}$, $t \in [0, 2\pi)$ up to some multiple of 2π

c) We find the Jacobian of f_1 is the gradient transpose, and the Hessian is

$$H(f_1) = \begin{bmatrix} -a\cos s\cos t & a\sin s\sin t \\ a\sin s\sin t & -(b+a\cos s)\cos t \end{bmatrix}.$$

Then

$$|H(f_1)| = a\cos s\cos t\,(b+a\cos s)\cos t - a^2\sin^2 s\sin^2 t$$
$$= a\cos^2 t\cos s\,(b+a\cos s) - a^2\sin^2 s\sin^2 t$$

at $(0,0)$

$H_{(0,0)} > 0$, $f_{ss}(0,0) < 0$ $\Rightarrow$ local maximum

$H_{(0,\pi)} > 0$ $f_{ss}(0,\pi) > 0$ $\Rightarrow$ local minimum

$H(\pi,0) < 0$, $\Rightarrow$ saddle point

$H(\pi,\pi) < 0$, $\Rightarrow$ saddle point

13) If $f : \mathbb{R} \to \mathbb{R}^3$ and $|f(t)| = 1$ for every t, then $f'(t) \cdot f(t) = 0$.

Intuitively, we may reason as follows. Suppose $f'(t) \cdot f(t) \neq 0$ for some $t \in \mathbb{R}$. Without loss of generality, suppose $f'(t) \cdot f(t) > 0$ in particular. Then $f(t)$ is not a local minimum of f, nor is f constant, a contradiction (else small variation gives $f(t+\delta) > f(t)$).

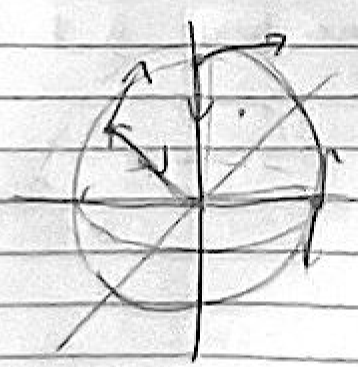
We see that

$$|f(t)| = \langle f(t), f(t) \rangle^{1/2} = 1$$
$$\Rightarrow \langle f(t), f(t) \rangle = 1$$

Differentiation then yields (noting $f(t), f'(t) \in \mathbb{R}^3$)

$$\frac{d}{dt}(1) = \frac{d}{dt} \langle f(t), f(t) \rangle = \langle f'(t), f(t) \rangle + \langle f(t), f'(t) \rangle$$
$$= 2 \langle f'(t), f(t) \rangle$$

$$\Rightarrow \langle f'(t), f(t) \rangle = f'(t) \cdot f(t) = 0.$$



E.g. on the sphere (l_2 norm) all such $f(t)$, $f'(t)$ have orthogonal components

Taking partials with x, y, z and u yields

$$\begin{bmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{bmatrix} = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z & \partial f_1/\partial u \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z & \partial f_2/\partial u \\ \partial f_3/\partial x & \partial f_3/\partial y & \partial f_3/\partial z & \partial f_3/\partial u \end{bmatrix}$$

Then taking determinants in x, y, z, u, we have

x: $$\begin{vmatrix} 1 & -1 & 2u \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix} = (4+3) + (-2-2) + 2u(3-4) = 7 - 4 - 2u = 3 - 2u$$

u: $$\begin{vmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} = 3(3-4) - (-3-4) - (2+2) = -3 + 7 - 4 = 0$$

y: $$\begin{vmatrix} 3 & -1 & 2u \\ 1 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix} = 3(4+3) + (2-2) + 2u(-3-4) = 21 - 14u$$

z: $$\begin{vmatrix} 3 & 1 & 2u \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 3(-4) - (0) + 2u(4) = -12 + 8u$$

So we see that we can solve in terms of $x, y,$ and z, but not u.

page 14 a) The sequence of matrices

$$\begin{pmatrix} 0 & 1/\delta & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_\delta, \ \delta \in \mathbb{N}$$

converges to the zero matrix. So the zero matrix is a boundary point of the non-diagonalizable matrices, hence the set of diagonalizable matrices is not open.

b) It is also not closed.

$$\begin{pmatrix} 1/\delta & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}_\delta, \ \delta \in \mathbb{N} \longrightarrow \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & \ddots \ 1 \end{pmatrix}$$

but this matrix is not diagonalizable

c) This set is not dense.

Take any matrix $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$. Consider the characteristic polynomial given by

$$\begin{vmatrix} x_1 - \lambda & y_1 \\ x_2 & y_2 - \lambda \end{vmatrix} = (x_1 - \lambda)(y_2 - \lambda) - x_2 y_1$$
$$= x_1 y_2 - \lambda y_2 - \lambda x_1 + \lambda^2 - x_2 y_1$$
$$= \lambda^2 - (x_1 + y_2)\lambda + x_1 y_2 - x_2 y_1$$

can be solved by

$$\lambda = \frac{(x_1 + y_1) \pm \sqrt{(x_1 + y_2)^2 - 4(x_1 y_2 - x_2 y_1)}}{2}$$

Now, if $(x_1+y_2)^2 - 4(x_1 y_2 - x_2 y_1) < 0$,
we have two complex eigenvalues.
each entry x_1, x_2, y_1, y_2 is continuous
and the above discriminant is a continuous
composition of continuous functions. So some
small δ-variations in each entries by which

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \longmapsto \begin{bmatrix} x_1+\delta_{x_1} & y_1+\delta_{y_1} \\ x_2+\delta_{x_2} & y_2+\delta_{y_2} \end{bmatrix}$$

gives a complex eigenvalue for some $\delta > 0$
and each terms δ-variation less than δ.
So the set is not dense in the set of
real valued matrices.

24. $f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^4}$

We have

$$\frac{\partial f(x,y)}{\partial x} = \frac{(3x^2y-y^3)}{(x^2+y^2)} + \frac{(x^3y+xy^3)(2x)}{-(x^2+y^2)^2}$$

$$= \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{(x^3-2xy)}{(x^2+y^2)} + \frac{(x^3y-xy^3)(2y)}{-(x^4+y^2)^2}$$

$$= \frac{x^5-4x^2y^2+x^3y^2}{(x^2+y^2)^2}$$

We see that taking derivative of each partial in the opposite coordinate yield respectively

$$\lim_{t\to 0} \frac{\frac{t^5}{t^4}}{t} = \frac{t^5}{t^5} = 1$$

$$\lim_{t\to 0} \frac{-\frac{t^5}{t^4}}{t} = -\frac{t^5}{t^5} = -1.$$

so

$$\frac{\partial f}{\partial x \partial y} = 1 \neq -1 = \frac{\partial f}{\partial y \partial x}.$$