Homework 5

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Exercise 1. *let $p > 2$ and $c > 0$. Then the set*

$$A = \{x \in [0,1] : |x - \frac{a}{q}| \leq \frac{c}{q^p} \text{ for infinitely many positive integers } a, q\}$$

has measure zero.

Proof. If $a > q$, then for such a, the set A is empty. So assume $0 \leq a \leq q$. For any $a, q \in \mathbb{Z}^+$, the inequality $|x - a/q| \leq c/q^p$ holds in the interval

$$[a/q - c/q^p,\ a/q + c/q^p].$$

Then for $0 \leq a \leq q$, for at most $q + 1$ intervals per q, summing over q and taking measure of intervals gives for each q, and corresponding set A_q

$$\sum_{q=1}^{\infty} m(A_q) \leq \sum_{q=1}^{\infty} \frac{2c(q+1)}{q^p}$$

which converges since $p > 2$ and therefore $\frac{c(q+1)}{q^p} \leq \frac{2c}{q^2}$. Then since $\sum_{q=1}^{\infty} m(A_q)$ has finite measure, Borel-Cantelli gives that A has measure zero.

□

Exercise 2. *For every positive integer n, let $f_n : \mathbb{R} \to [0, \infty)$ be a non-negative measurable function such that*

$$\int_{\mathbb{R}} \leq \frac{1}{4^n}.$$

For every $\epsilon \geq 0$, there exists a set E with Lebesgue measure $m(E) \leq \epsilon$ such that $f_n(x)$ converges pointwise to zero for all $x \in \mathbb{R} \setminus E$.

Proof. Let $\epsilon > 0$ and $A_n = \{x \in \mathbb{R} : f_n > \frac{1}{\epsilon 2^n}\}$. If $m(A_n) > \frac{\epsilon}{2^n}$ for some $n \in N$, then $\int_{\mathbb{R}} \geq \frac{1}{4^n}$ for this n. Otherwise take f_n to minorize g, then observe

$\int_{\mathbb{R}} g > \frac{1}{4^n}$. Taking union of the measurable A_n we have

$$m(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} \leq \epsilon.$$

and f_n converge pointwise on $R \setminus \bigcup_{n=1}^{\infty} A_n$. □

Exercise 3. *Let $f_n : \mathbb{R}^n \to [0, +\infty]$ be a sequence of measurable functions converging pointwise to $f = 0$. Then for every $\epsilon > 0$, there exists a subset $E \subset \mathbb{R}^n$ such that $m(E) < \epsilon$ and the f_n converge uniformly to f on each bounded $F \subset R^n \setminus E$.*

Proof. Let $f_n : \mathbb{R}^d \to [0, +\infty]$ be a sequence of measurable functions converging pointwise to f. Then for each $x \in R^d$, and each $\delta > 0$, there exists some $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \delta$ when $n \geq N$.

Let $U_{n,k} = \{x \in \mathbb{R}^d : |f_n(x) - f(x)| < 1/k\}$. For fixed k, $\bigcup_{n=1}^{\infty} U_{n,k} = \mathbb{R}^d$. Let (V_j) be a collection of bounded monotonically increasing measurable subsets of $\mathbb{R}^d$, such that $V_j \subset V_{j+1}$, and $\bigcup_{k=1}^{\infty} V_j = \mathbb{R}^d$. We see that $V_j = V_j \cap \bigcup_{n=1}^{\infty} U_{n,k}$ is measurable as each f_n is measurable, and the convergence holds as seen by considering each containment $V_j \cap \bigcup_{n=1}^{m} U_{n,k} \subset V_h \cap \bigcup_{n=1}^{m+1} U_{n,k}$. For any $\epsilon < 0$ and N sufficiently large, we may choose $n_k > N$ so that

$$m(V_j \setminus \bigcup_{n=1}^{n_k} U_{n,k}) = m(V_j \setminus U_{n_k}) < \epsilon/2^k,$$

and $|f_{n_k,k}(x) - f(x)| < 1/k$ on $V_j \cap U_{n_k}$. Letting $k \to \infty$ and taking $U = \bigcup_{k=1} {}_\infty U_{n_k}$ we see that the f is uniformly continuous on $V_j \cap U$ and $m(V_j \setminus U) \leq \sum_{k=1}^{\infty} m(V_j \setminus U_{n_k}) < \epsilon$. That is, the function is uniformly continuous on V_j and defining f by $f(x) = 0$ for all x, the result follows on the bounded set $[0, 1]$. For any ϵ' we may let $\epsilon = \epsilon'/2^j$ and sum over j and V_j to see the result holds on $\mathbb{R}^d$.

If we don't require F to be bounded, the statement doesn't necessarily hold, as the f_n could converge f pointwise but not uniformly.

□