Exercise 8.3.2. Let $\Omega$ be a measurable set and let $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ be absolutely integrable functions.

a) $\forall c \in \mathbb{R}$ $cf$ is absolutely integrable, and $\int_\Omega cf = c\int_\Omega f$.

Pf. $\int_\Omega |cf| = |c| \int_\Omega |f|$ as $f$ is absolutely integrable and following from properties of Lebesgue integrals of non-negative functions. Then since $\int_\Omega |f| < \infty$, we have $|c| \int_\Omega |f| < \infty$, and therefore $cf$ is absolutely integrable.

$\int_\Omega cf = \int_\Omega (cf)^+ - \int_\Omega (cf)^-$. If $c > 0$, this equals

$$\int_\Omega (cf)^+ - \int_\Omega (cf)^- = c\int_\Omega f^+ - c\int_\Omega f^-$$

$$= c\left(\int_\Omega f^+ - \int_\Omega f^-\right) = c\int_\Omega f.$$

If $c = 0$, $\int_\Omega cf = 0 = c\int_\Omega f$.

If $c < 0$, $\int_\Omega cf = \int_\Omega (cf)^+ - \int_\Omega (cf)^-$

$$= \int_\Omega (-c) f^- - \int c f^+ = c\int f^+ - c\int f^-$$

$$= c\left(\int_\Omega f^+ - \int_\Omega f^-\right) = c\int_\Omega f.$$

So $\int_\Omega cf = c\int_\Omega f$.

b) $f+g$ is absolutely integrable, and $\int_\Omega f+g = \int_\Omega f + \int_\Omega g$.

Pf. $\int_\Omega |f+g| \leq \int_\Omega |f| + \int_\Omega |g|$, so $\int_\Omega |f+g|$ is absolutely integrable.

Also, $f+g = (f^+ - f^-) + (g^+ - g^-) = (f+g)^+ - (f+g)^-$,

so $$\int_\Omega f+g = \int_\Omega (f+g)^+ - \int_\Omega (f+g)^-$$

$$= \int_\Omega f^+ - f^- + g^+ - g^-$$

$$= \int_\Omega f^+ - \int_\Omega f^- + \int_\Omega g^+ - \int_\Omega g^-$$

$$= \int_\Omega f + \int_\Omega g.$$

c) If $f(x) \le g(x)$ for all $x \in \Omega$, then $\int_\Omega f \le \int_\Omega g$.

Pf. $\int_\Omega f = \int_\Omega f^+ - \int_\Omega f^- \le \int_\Omega g^+ - \int_\Omega g^- = \int_\Omega g$.

since $f(x) \le g(x)$ $\forall x \in \Omega$.

d) $f(x) = g(x)$ for a.e. $x \in \Omega \Rightarrow \int_\Omega f = \int_\Omega g$.

Pf If $f(x) = g(x)$ on $\Omega \setminus E$ and $f(x) \neq g(x)$ on $E$, and $m(E) = 0$, then

$$\int_\Omega f = \int_{\Omega\setminus E} f + \int_{\Omega\cap E} f = \int_{\Omega\setminus E} f^+ - \int_{\Omega\setminus E} f^- + \int_{\Omega\cap E} f^+ - \int_{\Omega\cap E} f^-$$

$$= \int_{\Omega\setminus E} g^+ - \int_{\Omega\setminus E} g^- = \int_{\Omega\setminus E} g = \int_\Omega g.$$

Exercise 8.3.3 Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ be absolutely integrable, measurable functions s.t. $f(x) \leq g(x)$ $\forall x \in \mathbb{R}$. Then $f(x) = g(x)$ for a.e. $x \in \mathbb{R}$, iff $\int f(x) = \int g(x)$.

Pf. Let $E$ be the set $\{x \in \mathbb{R} : f(x) \neq g(x)\}$.

Then

$$\int_{\mathbb{R}} f = \int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^- = \int_{\mathbb{R}\setminus E} f^+ - \int_{\mathbb{R}\setminus E} f^- + \int_{\mathbb{R}\cap E} f^+ - \int_{\mathbb{R}\cap E} f^-$$

$$= \int_{\mathbb{R}} g = \int_{\mathbb{R}\setminus E} g^+ - \int_{\mathbb{R}\setminus E} g^- + \int_{\mathbb{R}\cap E} g^+ - \int_{\mathbb{R}\cap E} g^-.$$

Thus

$$\int_{\mathbb{R}\setminus E} f^+ - \int_{\mathbb{R}\setminus E} f^- = \int_{\mathbb{R}\setminus E} g^+ - \int_{\mathbb{R}\setminus E} g^-,$$

and if $m(\mathbb{R}\cap E) > 0$, then $\int_{\mathbb{R}} f < \int_{\mathbb{R}} g$

since $f < g$ on $\mathbb{R} \cap E$. Thus $m(\mathbb{R}\cap E) = m(E) = 0$.
so $f(x) = g(x)$ for almost every $x \in \mathbb{R}$.