Homework 7

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Exercise 1. *Suppose f and g are integrable and their squares are integrable. Then fg is measurable, integrable, and*

$$\int fg \leq \sqrt{\int f^2}\sqrt{\int g^2}.$$

Let $t \in R$. Then by linearity of Lebesgue integrals,

$$0 \leq p(t) := \int (f - tg)^2 = \int (f^2 - 2tfg + t^2 g^2) = \int f^2 + -2t \int fg + t^2 \int g^2$$

and

$$4((\int fg)^2 - \int f^2 \int g^2) \leq 0.$$

Since $p(t) > 0$ we see that

$$\int fg \leq \sqrt{\int f^2}\sqrt{\int g^2}.$$

Exercise 2.

Each $x \in [0, 1]$ may be expressed in base 3 as $(.\omega_1\omega_2\omega_3...)_3$. Then

$$x = \sum_{i=1}^{\infty} \frac{\omega_i}{3^i}$$

and each ω_i equals 0, 1, or 2. We've previously constructed a Cantor set C in which each $x \in C$ has a unique ternary expansion such that each ω_i equals 0 or 2. We define a function H on $[0, 1]$ by

$$H(x) = \sum_{1}^{\infty} \frac{\omega_i/2}{2^i}$$

where H has equals values at the endpoints of discarded gap intervals and we extend H to these intervals by letting H be constant on them. We recall that this function is well-defined and that it is continuous, surjective, and has derivative zero almost everywhere on $[0,1]$.

Define $\hat{H}:\mathbb{R}\to\mathbb{R}$ by $\hat{H}(x+n)=H(x)+n$ for all $n\in\mathbb{Z}$, $x\in[0,1]$. Define

$$H_k(x)=\hat{H}(3^k x) \quad J(x)=\sum_{k=0}^{\infty}\frac{\hat{H}(3^k x)}{4^k}.$$

For $x\in[0,1]$, $H(x)\le 1$ for $\in[0,1]$. Then for $x\in[0,1]$, for each k we have $\hat{H}(3^k x)\le 3^k$. Then $\sum_{k=0}^{\infty}\frac{\hat{H}(3^k x)}{4^k}$ converges. Since $\hat{H}(3^k x_0)<\hat{H}(3^k x_1)$ if $|x_0-x_1|>1/3^k$, summing over k we see that J is strictly increasing.

Recall that if a sequence of real valued functions defined on a set X satisfies $|f_n(x)|<M_n$ for all n, and all $x\in X$, where $M_n>0$, and $\sum_{n=1}^{\infty}M_n=L<\infty$, then $\sum_{n=1}^{\infty}f_n(x)$ converges absolutely and uniformly on X. We see then that $\sum_{k=0}^{\infty}\frac{\hat{H}(3^k x)}{4^k}$ converges on $[0,1]$. Since $\hat{H}(x+1)=H(x)+1$ for all x, J is continuous on $\mathbb{R}$.

We see that $J'=0$ almost everywhere. Fix $a>0$ and let

$$S_a=\{x: J' \text{ exists},\ J'>a,\ x \text{ belongs to the constancy intervals of every } H_k\}.$$

For each $N\in\mathbb{N}$ and any $x\in[0,\ 1/3^N]$, we define $J_N:[0,1/3^N]\to\mathbb{R}$ by

$$J_N(x)=\sum_{k=0}^{N}\frac{\hat{H}(3^k x)}{4^k}.$$

J_N is almost everywhere constancy intervals by the following reasoning. If $x\in[0,1]$ then $\hat{H}(x)=H(x)$, and $H(x)$ is almost everywhere constant (C has measure 0). For $x\in[0,\ 1/3^N]$ and $k<N$, we have $0\le 3^k x\le 1$. Then J_N is almost everywhere constancy intervals for such x. We next utilize the following theorem.

Fubini's differentiation theorem Let $I\subset R$ be an interval, and (f_k) a sequence of functions $f_k:I\to\mathbb{R}$. If $s(x):=\sum_{k=1}^{\infty}f(x)$ exists for all $x\in I$, and for every $k\in N$ f_k is an increasing function, then $s'(x)=\sum_{k=1}^{\infty}f_k'(x)$.

By the construction of H, we have for $x\in(0,\frac{1}{3^N})$ and $n\in\mathbb{N}$ that

$$H(x+\frac{n}{3^N})=H(x+\frac{3^N-n-1}{3^N})$$

and $H(x) = 1 - H(1-x)$ for $x \in [0,1]$. Since the sequence of functions (J_N) meets the hypotheses of Fubini's differentiation theorem, and since $f'_k = 0$ almost everywhere we have that $J'(x) = 0$ almost everywhere on $[0,1]$. Since $J(x)$ is defined in any intervals $[a, a+1]$ and $[a+1, a+2]$ identically up to a constant, the theorem implies that for almost every $x \in \mathbb{R}$, $J'(x) = 0$.

Exercise 3.

$$\int f(x,y)dx = \int_0^y \frac{1}{y^2}dx - \int_y^1 \frac{1}{x^2}dx = \frac{1}{y} + 1 - \frac{1}{y} = 1$$

$$\int f(x,y)dy = \int_1^x \frac{1}{y^2}dy - \int_0^x \frac{1}{x^2}dy = \frac{1}{x} - 1 - \frac{1}{x} = -1$$

yet double integral gives

$$\int_0^1 \int_0^x -\frac{1}{x^2}dydx = \int_0^1 \frac{1}{x}dx = \infty$$

so the integral does not exist. This does not contradict corollary 43, because this corollary requires the integrand to be non-negative.

Exercise 4.

a) Since E is measurable, this is just a restriction of the density of some point of E, and the conclusion follows.

b) Let $A = \{x \in \mathbb{R} : \sin(1/x) > 0\}$, and $B = \mathbb{R} \setminus A$. Then on the any interval $(-r, r)$, centered at 0, $m(B \cap (-r, 0)) = -m(A \cap (0, r))$, as $B \cap (-r, 0) = A \cap (0, r) - 2r$, that is, the opposite sets are obtained by complement in a interval of length r, and measure preserving translation. Then centered at 0, we have balanced density 1/2. It seems that some iterated addition or subtraction of additional periodic functions could change the density. In R^2, the solution is more clear.

Exercise 5.

Take some convergent strictly increasing series and allow it to attain its subsequent sums on $\mathbb{Q}$. So if $\sum_{n=1}^{\infty} a_n \to L < \infty$, the terms of $a_n \to 0$. For a given bijection $f : \mathbb{N} \to \mathbb{Q} \cap [0,1]$, we can take the sum

$$s(x) = \sum_{n \in E_x} a_n$$

where $E = \{n \in \mathbb{N} : f(n) < x\}$.

Then the "partial sum" function $s(x)$ is monotonically increasing, and its discontinuities are at the points $\mathbb{Q} \cap [0, 1]$. Because the sum $s(x)$ is convergent, for any $p \in [0, 1]$, we have that $f(x) \to s(p)$ as $x \to p$, so the discontinuity points are just those in $Q \cap [0, 1]$.