Homework 8

Samuel Speas

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1 Introduction

Exercise 1.

a) We prove Egorov's theorem in homework 5.

b) This is true, as shown in homework 5.

c) The "moving bump" example, where $k > 0$ and $f_n := 1_{[n,n+k]}$ on R is such a sequence. This sequences converges pointwise to the zero function on R, but for ϵ such that $0 < \epsilon < 1$ we have that $|f_n(x) - f(x)| > \epsilon$ on a set with measure k.

d) One way to see that this is true is to take the V_j in the proof of Egorov's theorem large enough so that $K \subset V_j$, by choosing large enough j.

Alternatively, let $C = \bigcup_{i=1}^N Q_i$ for compact sets Q_i, where N and Q_i are chosen so that $\bigcup_{i=1}^N Q_i$ covers K. For each i Egorov's theorem allows us to find closed subsets $S_i \subset Q_i$ such that $m(S_i) < \epsilon/2^i$ where ϵ is given, and the sequence (f_n) converges uniformly on $Q_i \setminus S_i$. Let $S = \bigcup_{i=1}^N S_i$. Then

$$K \cap S \subseteq C \cap S$$

and therefore

$$m(K \cap S) \leq m(C \cap S) \leq \sum_{i=1}^{N} m(S_i) \leq \sum_{i=1}^{N} \frac{\epsilon}{2^i} < \epsilon.$$

Thus we have shown that for each compact $K \subset \mathbb{R}^n$, the sequence of functions (f_n) converges uniformly on $K \setminus S$, where $m(S) < \epsilon$.

Exercise 2.

Let T_j denote the jth column of matrix T. Then

$$|Tx|_1 = |\sum_{j=1}^{n} x_j(T_j)|_1 \le \sum_{j=1}^{n} |x_j(T_j)|_1 = \sum_{j=1}^{n} |x_j||T_j|_1$$

$$\le \left(\sum_{j=1}^{n} |x_j| \right) \max_{1 \le j \le n} \{|T_j|_1\} = |x|_1 \max_{1 \le j \le n} \{|T_j|_1\}.$$

So we have

$$\frac{|Tx|_1}{|x|_1} \le \max_{1 \le j \le n} \{|T_j|_1\}.$$

Suppose that column k is such that $\max_{1 \le j \le n}\{|T_k|_1\} = |T_k|_1$. Then let $c \in \mathbb{R}$ and $x = e_k$, and we see that

$$\max_{1 \le j \le n} \{|T_j|_1\} \le \frac{|T_k x|_1}{|x|_1}.$$

Then since

$$\max_{1 \le j \le n} \{|T_j|_1\} \le \frac{|T_k x|_1}{|x|_1} \le \max_{1 \le j \le n} \{|T_j|_1\}$$

we have

$$|T|_1 = \max_{1 \le j \le n} \{|T_j|_1\}.$$

Exercise 3.

Holder and Minkowski Inequalities: Holder's inequality follows from Young's inequality and properties of measurable functions. Young's inequality follows from convexity.

We can use properties of convexity and Holder's inequality to prove Minkowski inequality.

Littlewood's three principles

(i) Every (measurable) set is nearly a finite sum of intervals;

(ii) Every (absolutely integrable) function is nearly continuous;

(iii) Every (pointwise) convergent subsequence of functions is nearly uniformly convergent.