(1) 6. Let $f(x,y) = \frac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$

$f(0,0) = 0$

The functions xy and x^2+y^2 are differentiable and when $(x,y) \neq (0,0)$, we have

$$\frac{\partial f}{\partial x} = \frac{-x^2y+y^3}{(x^2+y^2)^2}, \quad \frac{\partial f}{\partial y} = \frac{x^3-xy^2}{(x^2+y^2)^2}$$

If $(x,y) = (0,0)$, we have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-0}{t} = \left(\frac{0}{t^2} - 0\right)/t = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-0}{t} = \left(\frac{0}{t^2} - 0\right)/t = 0.$$

So $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere in $\mathbb{R}^2$.

6. continued

f is not continuous at $(0,0)$:

$$\lim_{t \to 0} f(t,t) = \lim_{t \to 0} \frac{t^2}{2t^2} = \frac{1}{2}.$$

yet $f(0,0) = 0$, so f is not continuous at $(0,0)$.

(2) 7. Suppose f is a real-valued function defined on open set $E \subset \mathbb{R}^n$, and its partial derivatives are bounded in E.

Let $M = \max \left\{ \left\| \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) \right\| : \begin{array}{l} 1 \le i \le n, \\ (x_1, \ldots, x_n) \in E \end{array} \right\}$

Given any $a, b \in E$, we have that $b = a + (b-a)$, so we let $c = b - a$, so that $b = a + c$, and $c = (c_1, \ldots, c_n)$

Let $b_k = a + \sum_{i=1}^{k} c_i e_i$, $b_0 = a$ for $0 \le k \le n$

We may draw a path $\sigma = [\sigma_1, \ldots \sigma_n]$ from a to b where $\sigma_k = b_{k-1} + t c_k$ $(0 \le t \le 1)$ is a segment from b_{k-1} to b_k.

7. cont. Define $g(t) = f \circ \sigma_k(t)$. By the chain rule and mean value theorem for differentiable single variable functions there exists $t_k \in (0,1)$ such that

$$f(b_k) - f(b_{k-1}) = g(1) - g(0) = \frac{\partial f(\sigma(t_k))}{\partial x_k} c_k$$

Then $|f(b) - f(a)| = |f(a+c) - f(a)|$

$$= \left|\sum_{k=1}^{n} (f(b_k) - f(b_{k-1})\right|$$

$$\leq \sum_{k=1}^{n} \left|\frac{\partial f(\sigma(t_k))}{\partial x_k}\right| |c_k|$$

$$\leq \sum_{k=1}^{n} M|c_k| \leq nM \cdot |c|_\infty$$

Given $\varepsilon > 0$,
let $\|a-b\| < \varepsilon/nM$. Then $|f(b) - f(a)| < \varepsilon$, so f is continuous.

(3) Let $E \subset \mathbb{R}^2$. Define $f: \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in E \\ e^{-\frac{1}{g(x)}} & \text{if } x \notin E \end{cases}$$

where $g(x) = \inf\{|x-y| : y \in E\}$

Then $f^{-1}(0) = E$, and is continuous, and smooth.

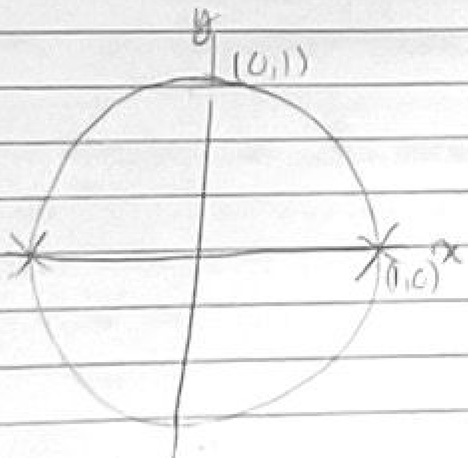
(4) If $n = m = 1$ for the implicit function theorem we have a function $f: \mathbb{R}^2 \to \mathbb{R}$, and the theorem says that if $(x_0, y_0) \in \mathbb{R}^2$ and

if $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$ then if $f(x_0, y_0) = z_0 \in \mathbb{R}$

then the z_0-locus near (x_0, y_0) is a graph $y = g(x)$.

Consider the function $f(x, y) = x^2 + y^2$. Then since $\frac{\partial f}{\partial y} = 2y$, if $y_0 \neq 0$ then

at $f(x_0, y_0)$ we may apply the theorem and find that we may express y as a function of x.



We have $y = g(x)$ for some g, when $y \neq 0$.