Homework 4

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Exercise 1 (a)**.** *Graphs of measurable functions have measure zero.*

We reason in the in the spirit of homework 4, exercise 1, but utilize the strength of Lebesgue measure and the definition of measurable functions rather than grid partitions. Let $E \subset \mathbb{R}$ where $m(E) < \infty$ and f be an unsigned measurable function $f : E \to \mathbb{R}^+$.

For fixed $n \in N$ We define the sets

$$E_j = \{x \in E : \frac{j}{n} \leq f(x) < \frac{j+1}{n}\}$$

and the countable collection $(E_j)_{j \in J}$. We then see that for graph

$$g = \{(x, f(x) : x \in E\}$$

of f that

$$g = \bigcup_{j=0}^{\infty} \{f(x) : x \in E_j\}.$$

Then for we have for each E_j

$$m(g) \leq \sum_{j=1}^{\infty} \frac{1}{n} m(E_j).$$

Letting $n \to \infty$ we have that $m(g) = 0$. If instead we have $f : F \to \mathbb{R}$ and F is not bounded, partition F into sets given in countable collection (E^i) and for given > 0 let $1/n_i$ be such that $\sum_i^{\infty} m(\bigcup E^i) <$. Let $\to \infty$ and the result follows for unbounded F.

Exercise 2 (b)**.** *If function f has graph measure 0, it is not necessarily the case that f is measurable*

We see that if $E \subset \mathbb{R}$ is not measurable, then for

$$f = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

the graph of f has measure zero but f is not measurable.

Exercise 3 (c)**.** *See https://math.stackexchange.com/questions/593389/non-measurable-set-a-such-that-every-measurable-subset-of-a-or-ac-has-mea*

Exercise 4 (d)**.** *Without the measurability hypothesis in the zero slice theorem, we can construct with transfinite induction, a function such that almost every slice has measure zero, yet has positive outer measure*

Exercise 5 (e)**.** *Graphs can't have positive inner measure*

Otherwise some vertical slice has positive outer measure, which is clearly not possible since graphs contain only a single point along a given dimension.

Exercise 6 (f)**.** *Translations*

If the graph has inner measure zero, we can translate it into uncountable many disjoint subets of the plane with positive outer measure since each real translate has positive outer measure

Exercise 7 (a)**.** *The total undergraph $\underline{U}f$ of $f : R \to R$ is measurable if and only if $\underline{U}f \cap \mathbb{R}^+$ and $\underline{U}f \cap \mathbb{R}^-$ are measurable.*

The intersection of finitely many measurable sets is measurable, as is the finite union.

Exercise 8 (b)**.** *If $f : \mathbb{R} \to \mathbb{R}^+$ is measurable, then $1/f$ is measurable.*

f is measurable ($\mathcal{U}f$ is measurable) and $T : (x, y) \to (x, 1/y)$ is a diffeomorphism that sends $\mathcal{U}f$ to $\mathcal{U}1/f$, as if $(x, y) \in U1/f$, then $T(x, y) = (x, 1/y) \in \mathcal{U}1/f$ and if $(x, y) \in \mathcal{U}1/f$, $(x, 1/y) \in \mathcal{U}f$. Diffeomorphisms are meseomorphisms, so $\mathcal{U}1/f$ is measurable, and $1/f$ is measurable.

Exercise 9 (c)**.** *if $f, g : \mathbb{R} \to \mathbb{R}^+$ are measurable functions, then $fg : \mathbb{R} \to \mathbb{R}+$ is a measurable function.*

Since $\log(x)$ and e^x are differentiable on $(0, +\infty$, they are diffeomorphisms and hence meseomorphisms on $(0, +\infty)$. Then for he functions f, g, we have $\mathcal{U}(\log(g(x))$ and $\mathcal{U}(\log(f(x))$ are measurable. Then since f, g are measurable, $f(x) + g(x) = \log(f(x)g(x))$. Then $f(x)g(x) = e^{\log(f(x)g(x))}$ is also measurable.

Exercise 10 (e)**.** *Remove the hypothesis that the domain is R*

Assuming we then have domain $\Omega \subset \mathbb{R}$, the results still hold by intersection of given undergraphs with $\{(a, b) : a \in \Omega, b \in \mathbb{R}\}$.