1/2 the course: Lebesgue measure
2/2 : - Multivariable Calculus + Stoke's Theorem
- Fourier Analysis

Professor Role:
- lecturer: Actor
- Coach
- Tour guide

Grade: 50% participation 50 % Final
- Share notes
- Homework solution

To Do
- ☐ Read Syllabus
- ☐ Download textbooks
- ☐ Upload notes

Shortcomings of the Riemann Integral:
- The underlying space is like $\mathbb{R}^n$;
- we can only consider bounded
- If $f_n \to f$ pointwise and f_n is Riemann integrable, it is not true that f is Riemann integrable.

Enter the **Lebesgue integral**:

$$\int_\Omega f\,dx \quad \text{where } \Omega \subset \mathbb{R}^n, \quad dx = dx_1 \cdots dx_n$$

- What Ω do we allow?
  - Lebesgue measurable sets
- What f do we allow?
  - Lebesgue integrable functions

The **Lebesgue Measure**: $m(\Omega) = \int_\Omega 1\,dx$

- If $\Omega \subset \mathbb{R}$, then $m(\Omega)$ should be the <u>length</u> of Ω
- If $\Omega \subset \mathbb{R}^n$, then $m(\Omega)$ should be the <u>volume</u> of Ω
- We know the length of $[a,b]$: $|b-a|$ But can we "consistently" define "length" for any subset of $\mathbb{R}^n$?
  - **The desireable properties:**

    1) monotonicity : $A \subset B \subset \mathbb{R}^n \rightsquigarrow m(A) \leq m(B)$

    2) additivity : If $A \cap B = \emptyset$ then $m(A \cup B) = m(A) + m(B)$

Q: If $A \subsetneq B \subset \mathbb{R}^n$, when will we have $m(A) = m(B)$?

examples:
- $(a,b) \subset [a,b]$
- $\{1\} \subset \{1,2\}$

3) translation invariance: $\forall x \in \mathbb{R}^n$, $E \subset \mathbb{R}$, we want $m(E) = m(x+E)$

But there is a problem: m does not exist! (Read: Tao 7.3 or Wiki: The Banach-Tarski paradox)

We get around this problem by restricting the class of subsets in $\mathbb{R}^n$ to which we assign measure.

Axioms of Measureable Subsets (Tao II, pg. 180)

notation: If S is a set, 2^S denotes the set of subsets of S.

Let M_n denote the set of measureable subsets of $\mathbb{R}^n$.

(1) If $U \subseteq \mathbb{R}^n$ is open, then $U \in M_n$

(2) If $U \in M_n$, $U^c \in M_n$

(3) If $U, V \in M_n$, then $U \cup V \in M_n$ and $U \cap V \in M_n$

2 and 3 imply M_n is a Boolean algebra. However we want something stronger: A σ-algebra If $U_1, U_2, \ldots$ is a sequence of measurable sets, then $\bigcup_n U_n$ and $\bigcap_n U_n$ are both measureable.

Axioms of the Lebesgue Measure: (Read Tao !!!)

Thm: There exists a definition of M_n and Lebesgue measure satisfying the axioms.

Def^n: Outer Measure

$\forall E \subset \mathbb{R}^n$, define countably infinite!

$$m^*(E) = \inf \{ \sum_{i=1}^{\infty} |B_i|, \; E \subset \bigcup_{i=1}^{\infty} B_i, \; B_i \subset \mathbb{R}^n \text{ are open boxes} \}$$

m^* is well-defined for all subsets.

In discussion: Try to prove nice properties of m^*... (Lemmas 7.2.5 and 7.2.6)

Positivity: inf of $\sum^{\infty} |\text{positive}|$ is never negative.