- Homework 7 is up.

Last lecture we defined <u>outer measure</u> m^*

Today's goal: work out properties of m^* (7.2.5)

- Empty set: No cover is needed
- Nonnegativity: inf over nonnegatives
- Monotonicity: For any open cover of B, if $A \subset B$, the cover covers A as well. So

$$\inf\{\Sigma \text{ cover of } A\} \le \inf\{\Sigma \text{ cover of } B\}$$

since volume covering $A \le$ volume covering B

- Finite sub-additivity: $m^*(A \cup B) \le m^*(A) + m^*(B)$

Try proving $m^*(A) + m^*(B) \ge$ (area of some covering of A, and covering of B) $- \varepsilon$

then total area $\ge m^*(A \cup B)$ thus

$\forall \varepsilon > 0 \quad m^*(A) + m^*(B) \ge m^*(A \cup B) - \varepsilon \Rightarrow m^*(A) + m^*(B) \ge m^*(A \cup B)$

$\because m^*(A) = \inf\{\Sigma \operatorname{vol}(B_i) \mid \{B_i\} \text{ cover } A\}$

$\therefore \forall \varepsilon > 0 \quad \exists$ covering $\{B_i\}$ s.t. $\Sigma_i |B_i| \le m^*(A) + \varepsilon$

similarly do it for B. Then take the union of the two covers to get a cover of $A \cup B$

- Countable sub-additivity.

W.T.S.: $m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \le \Sigma\, m^*(A_j)$

$\forall \varepsilon > 0 \quad \exists$ covers $\{B_i^{(j)}\}$ for A_j s.t.

$$m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_j m^*(A_j) + \varepsilon$$

$$= \sum_{j=1}^{\infty}\left(m^*(A_j) + \frac{\varepsilon}{2^j}\right)$$

We can find open cover $\{B_i^{(j)}\}$ for A_j s.t. ...?

Prop^n:

Outer measure of closed box B:

$$m^*(B) = \prod_{i=1}^{n} (b_i - a_i)$$

(Compare to the Riemann integral)

proof.

$\forall \varepsilon > 0$ we can choose a open box slightly larger than B to cover B, thus

$$m^*(B) \leq vol(B) + \varepsilon$$

(n=1 case)

Because $B=[a,b]$ is compact any open cover of B can be reduced to a finite subcover. Let $\{B_i\}_{i=1}^N$ be a finite open cover of B. We W.T.S.

$$\sum_{i=1}^{N} |B_i| \geq vol(B)$$

Let $f_i(x) = 1_{B_i}(x)$, then $\sum_{i=1}^{n} |B_i| = \sum_{i=1}^{n} \left(\int_{\mathbb{R}} f_i \, dx \right) = \int_{\mathbb{R}} \underbrace{\sum_{i=1}^{N} f_i}_{f(x)} \, dx$

<u>Claim:</u> $f(x) \geq 1_B(x)$.

Indeed $B \subset \bigcup_{i=1}^{N} B_i$ thus $1_B \leq \sum 1_{B_i}$

(n=2 case)

WTS given any finite cover $\{B_i\}_{i=1}^N$ of B that $\sum_{i=1}^{N} |B_i| \geq |B|$.

Again $|B_i| = \int_{\mathbb{R}^2} 1_{B_i}(x_1, x_2) \, dx_1 \, dx_2 = \int_{\mathbb{R}} w_i \cdot 1_{B_{i,1}}(x_1) \, dx_1$

$w_i \{ \boxed{B_i}$

Integrating along x_2:

$$\sum_{i=1}^{N} \int 1_{B_i}(x) \, dx_1 \, dx_2 = \int_{\mathbb{R}^2} \sum_{i=1}^{N} 1_{B_i(x)} \, dx_1 \, dx_2$$

$$= \int_{\mathbb{R}} \left(\int \sum_i 1_{B_i}(x_1, x_2) \, dx_2 \right) dx_1$$

<u>Claim:</u> $f(x_1) \geq 1_{[a_1, b_1]}(x_1) \cdot \underbrace{|b_2 - a_2|}_{\text{height of } B}$

$\Leftrightarrow x_1 \in [a_1, b_1]$. $f(x_1) \geq |b_2 - a_2|$. If $x_2 \notin [a_2, b_2]$

$f(x_1) \geq 0$ is true

And

$$\int_{\mathbb{R}} \left(\int \sum_i 1_{B_i}(x_1, x_2)\, dx_2 \right) dx_1$$

$$\geq \int_{\mathbb{R}} 1_{[a_1, b_1]} (b_2 - a_2)\, dx_1$$

$$= (b_2 - a_2)(b_1 - a_1)$$

$$= \mathrm{vol}(B)$$