Recall:

$m^*(A) := \inf \left\{ \sum_i |B_i| ,\ \{B_i\} \text{ is a countable open cover of boxes of } A \right\}$

Properties:

- $m^*(\phi) = 0$
- $A \subseteq B \Rightarrow m^*(A) \leq m^*(B)$
- $A = \bigcup_{i=1}^{\infty} A_i \Rightarrow m^*(A) \leq \sum_{i=1}^{\infty} m^*(A_i)$

A set $E \subseteq \mathbb{R}^n$ is measurable iff $\forall A \subseteq \mathbb{R}^n$, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

Today:

Lemma 7.4.2: Half spaces are measurable.

$n=1$ case:

i.e. $(0,\infty)$ is measurable.

WTS: $\forall A \subseteq \mathbb{R}$,

$$m^*(A) = m^*(A \cap (0,\infty)) + m^*(A \cap (-\infty, 0])$$

Note: $A = (0,\infty) \sqcup (-\infty, 0]$ so by finite sub-additivity:

$$m^*(A) \leq m^*(A \cap (0,\infty)) + m^*(A \cap (-\infty, 0])$$

To show $m^*(A) \geq m^*(A \cap (0,\infty)) + m^*(A \cap (-\infty, 0])$:

Suffices to show:

$\forall \varepsilon > 0$

$$m^*(A) + \varepsilon \geq m^*(A \cap (0,\infty)) + m^*(A \cap (-\infty, 0]).$$

consider an open cover of A by open boxes $\{B_j\}$ such that $\sum |B_j| \leq m^*(A) + \varepsilon/2$

Define $B_j^+ = B_j \cap (0, \infty)$, $B_j^- = B_j \cap (-\infty, \varepsilon/2^{j+1})$

Then $B_j = B_j^- + B_j^+$ and

$$|B_j| + \frac{\varepsilon}{2^{j+1}} \geq |B_j^+| + |B_j^-| \geq |B_j|$$

Moreover, $\bigcup B_j^+ \supseteq A \cap (0,\infty)$, and $\bigcup B_j^- \supset A \cap (-\infty, 0]$

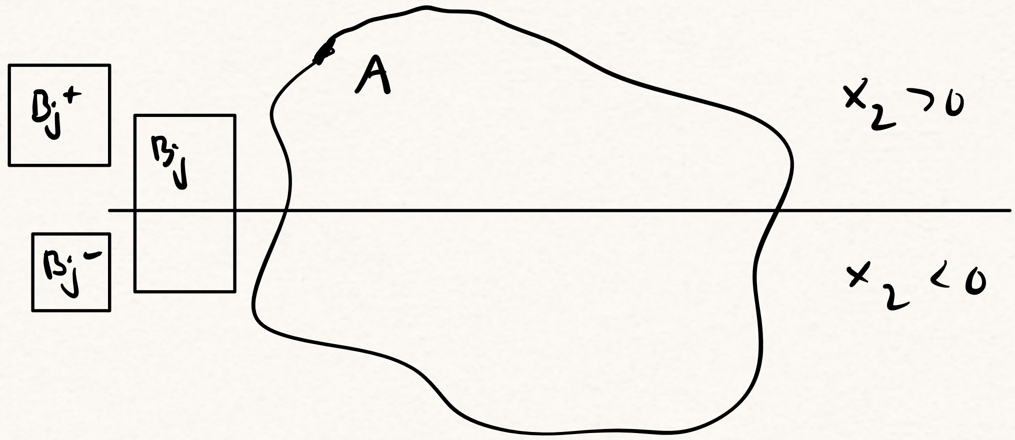
thus $m^*(A \cap (0,\infty)) + m^*(A \cap (-\infty, 0])$.

$$\begin{aligned}
&\leq \sum |B_j^+| + \sum |B_j^-| \\
&\leq \sum_{j=1}^{\infty} \left(|B_j| + \frac{\varepsilon}{2^{j+1}}\right) \\
&\leq \left(\sum |B_j|\right) + \frac{\varepsilon}{2} \\
&\leq m^*(A) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= m^*(A) + \varepsilon
\end{aligned}$$

Try $\mathbb{R}^n$; $n=2$ case...

Need to show

$$m^*(A) + \varepsilon \geq m^*(A \cap (0,\infty)) + m^*(A \cap (-\infty, 0])$$



Tao Ex 7.4.3:

For A an open box in $\mathbb{R}^n$, prove

$$m^*(A) = m^*(A^+) + m^*(A^-).$$

Should be easy, since $m^*(A) = |A|$,
$m^*(A^+) = |A^+|$...

For a general A, for any $\varepsilon > 0$, find $\{B_j\}$ cover of A s.t. $m^*(A) + \varepsilon \geq \sum |B_j|$.

Define $B_j^+ = B_j \cap \{x_n > 0\}$, $B_j^- \cap \{x_n < 0\}$
(They may not be open)

Since $A_+ \subseteq \bigcup B_j^+$ we have

$$m^*(A^+) \leq \sum m^*(B_j^+) = \sum |B_j^+|$$

Similarly:

$$m^*(A^-) \leq \sum m^*(B_j^-) = \sum |B_j^-| \quad \blacksquare$$

Lemma 7.4.4: Properties of measurable sets

(a) If $E \subseteq \mathbb{R}^n$ is measurable, then so is E^c

(b) Translation invariance

(c) If E_1, E_2 are measurable, then $E_1 \cap E_2$ and $E_1 \cup E_2$ are as well.

proof.

WTS:

$$m^*(A) = m^*(A \cap (E_1 \cap E_2)) + m^*(A \setminus (E_1 \cap E_2))$$

$$\vdots$$

(d) Boolean Algebra: Finite $\cup$ a $\cap$ preserves measurability.

(e) Every box, open, closed, or half open/half closed, is measurable.

(f) If $m^*(E) = 0$, then E is measurable.

Discussion : 7.4.5