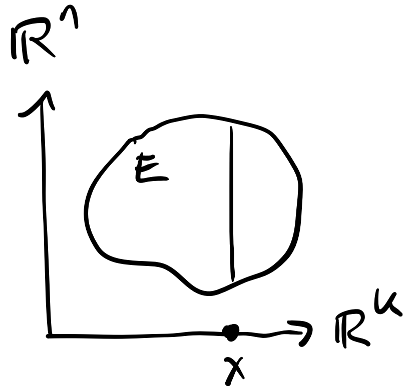
Today: • Finish slice theorem
• Lebesgue integral

Recall:



$E \subset \mathbb{R}^k \times \mathbb{R}^n$

$x \in \mathbb{R}^k$

$E_x = E \cap \{x\} \times \mathbb{R}^n \subset \mathbb{R}^n$

Thm:

$m(E) = 0$ iff almost every slice E_x has measure zero. i.e. if

$$Z = \{x \mid m(E_x) \overset{=?}{>} 0\}$$

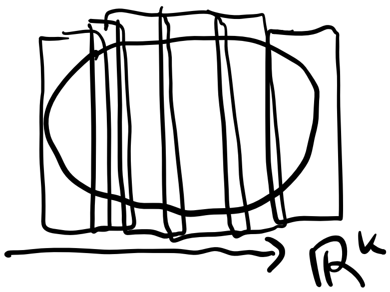
then $m(Z) = 0$, $Z \subset \mathbb{R}^k$

Last time:

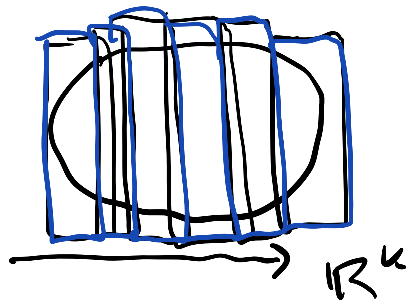
($\Leftarrow$) ① May assume $Z \neq \phi$, E is bounded

② Use inner approx. of E by compact subset K

③ Cover K by finitely many open boxes vertically long enough $U_i \times V_i$, then disjointize.



disjointize it $\rightsquigarrow$



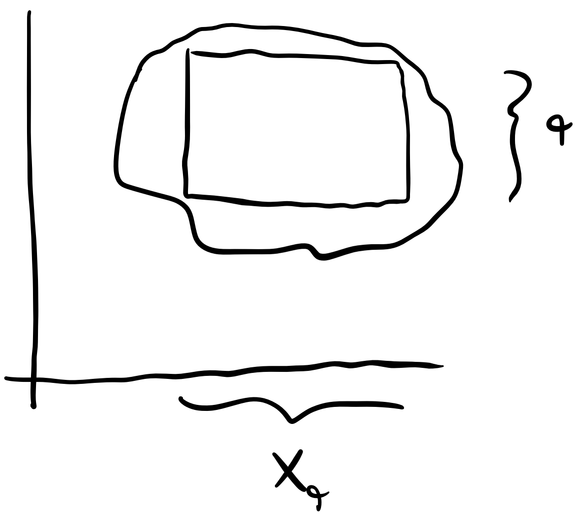
④ $m(K) \leq \sum \underset{\overset{\wedge}{1}}{m(U_i)} \times \underset{\overset{\wedge}{\varepsilon}}{m(V_i)} < \varepsilon$

⑤ $m(E) < m(K)$ ... ???

Lemma:

$\forall W$ open (bounded), $\forall \alpha > 0$, let $X_\alpha = \{x \in \mathbb{R}^k \mid m(W_x) > \alpha\}$

$W \subset \mathbb{R}^k \times \mathbb{R}^n$, then $m_{n+k}(W) \geq m_k(X_\alpha) \cdot \alpha$.



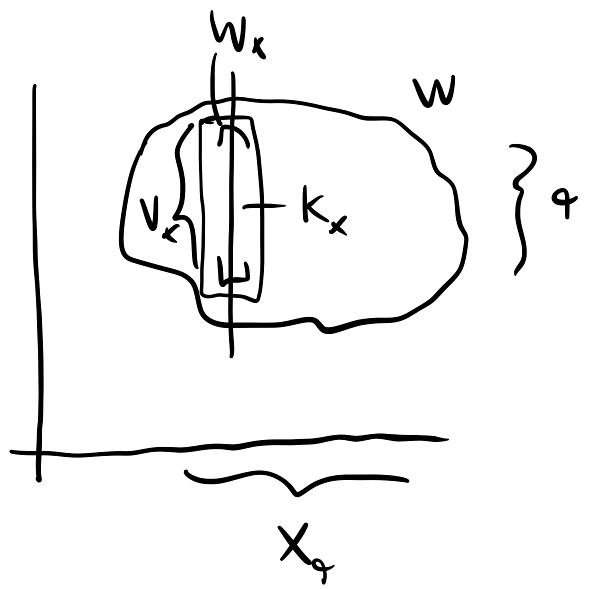
$m(W) \geq m(\square) = m(X_\alpha) \cdot \alpha$

proof.

① $\forall x \in X_q$, get a compact set $K_x \subset W_x$

$m(K_x) > q$

② Fatten K_x to an open box $\underset{\mathbb{R}^k}{U_x} \times \underset{\mathbb{R}^n}{V_x} \subset W$



$\Rightarrow X_q$ is open since $U_x \subset X_q$, indeed for all

$x' \in U_x$, $W_{x'} \supset U_{x'} \supset K_x$

so $m(W_x) \geq m(K_x) > q$

② $\forall K'$ compact in X_q

$\because X_q \subset \bigcup_{x \in X_q} U(x)$

$\because$ we have a finite subcover for K'

$K' = U(x_1) \cup \dots \cup U(x_n)$

$U_1 = U(x_1)$

$U_2 = U(x_2) \setminus U(x_1)$

$\vdots$

all U_i are disjoint

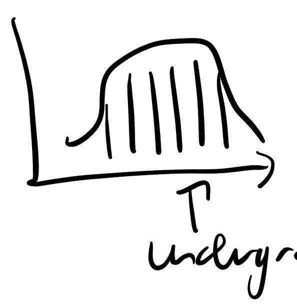
$$m(W) \geq m\left(\bigsqcup_{i=1}^{n} U_i \times V(x_i) \right) \geq \sum m(U_i) \cdot q$$

$$\geq m(K') \cdot q$$

<u>Pugh 6.6</u> Lebesgue Integral

Let $f: \mathbb{R} \to [0, \infty)$. We define the <u>undergraph</u> $U(f)$ by

$$U(f) = \{(x,y) \mid 0 \leq y < f(x)\}$$



We say f is <u>measurable</u> if $U(f)$ is a measurable subset. We define

$$\int f := m(U(f)) \quad (\text{possibly } +\infty)$$

and we say f is integrable if $m(U(f))$ is finite.

"Almost everywhere" means "up to a measure zero set."

e.g. $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$

then $f(x) = 0$ a.e.

Thm 27:

Let $f_n : \mathbb{R} \to [0,\infty)$ be a sequence of measurable functions and $f_n \nearrow f$ almost everywhere as $n \to \infty$ (i.e. $\exists$ a null set $Z \subset \mathbb{R}$ s.t. $\forall x \in \mathbb{R} \setminus Z$, $\lim_{n\to\infty} f_n(x) = f(x)$, $f_{n+1}(x) \geq f_n(x)$.) Then $\int f_n \nearrow \int f$

proof.

$f_n \nearrow f \Rightarrow U(f_n) \nearrow U(f)$ (i.e.

$A_n \nearrow A$ means $A_n \subset A_{n+1} \subset \cdots$, $A = \bigcup A_n$

Then $m(U(f_n)) \nearrow m(U(f))$ by countable additivity.

Def^n:

The <u>completed undergraph</u> $\hat{U}(f)$ is

$$\hat{U}(f) = \{(x,y) \mid 0 \le y \le f(x)\}$$

Prop^n:

$U(f)$ measurable $\iff$ $\hat{U}(f)$ measurable.

$$m(U(f)) = m(\hat{U}(f))$$

Fact: (Pugh 6.3)

If $T: \mathbb{R}^n \to \mathbb{R}^n$ is an affine lin transf.

i.e. $T(x) = Ax + b$

If $E \subset \mathbb{R}^n$ is measurable,

then

$$m(T(E)) = |T| \cdot m(E)$$

where $|T| = |\det A|$

proof.

($\Rightarrow$) $\forall n > 0$ integer,

$$U(f) \subset \hat{U}(f) \subset \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1+\frac{1}{n} \end{pmatrix} U(f) + \mathbb{R} \times \{0\}}_{R_n(f)}$$

then

$$\bigcap_{n=1}^{\infty} R_n(f) \text{ is meas.}$$

$$m\left(\bigcap R_n(f)\right) = \lim m(R_n(f))$$

$$\vdots$$

Prop^n: (29)

If f_n is a seq. of intgr. functions that $f_n \searrow f$ a.e., then $\int f_n \searrow \int f$

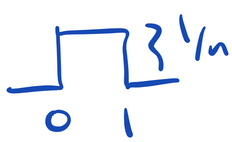
proof.

$$m(U(f)) = m(\hat{U}(f)) \swarrow m(\hat{U}(f)$$

$$= m(U(f_n))$$

$$= \int f_n < \infty$$

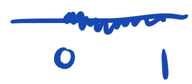
$U(f_n) \searrow U(f)$? No!

Suppose $f_n =$ 

$f = 0$

$U(f) = \phi$

$U(f_n)$



Recall:

If a_n is a bounded sequence,

$$\bar{a}_n := \sup\{a_m \mid m \geq n\}$$

$$\underline{a_n} := \inf\{a_m \mid m \geq n\}$$

Now if $f_n(x)$ is a sequence of functions, we define

$$\overline{f_n(x)} = \sup\{f_m(x) \mid m \geq n\}$$

$$\underline{f_n}(x) = \inf\{f_m(x) \mid m \geq n\}$$

Prop^{n}:

$$U(\bar{f}_n) = \bigcup_{k \geq n} U(f_k)$$

$$\hat{U}(\bar{f}_n) = \bigcap_{k \geq n} \hat{U}(f_k)$$

<u>Thm:</u>

Suppose we have a collection of meas. functions $f_n \to f$ a.e. and $\exists g : \mathbb{R} \to [0,\infty)$ s.t. $g(x) \geq f_n(x)$ a.e., and $\int g < \infty$. Then

$$\int f_n \to \int f$$

proof.

$\because \mathcal{U}(f_n) \subset \mathcal{U}(g)$.

$\therefore m(\mathcal{U}(f_n)) \leq m(\mathcal{U}(g)) < \infty$

- $\mathcal{U}(\underline{f_n}) \subset \mathcal{U}(f_n) \subset \hat{\mathcal{U}}(\overline{f_n})$
- $\underline{f_n} \nearrow f, \quad \overline{f_n} \searrow f,$

$$\int \underline{f_n} = \int f = \int \overline{f_n} = \lim \int \underline{f_n} = \lim \int \overline{f_n} = \lim \int f_n$$

<u>e.g.</u>

$$f_n(x) = 1_{[n,n+1]}(x).$$

$f_n(x) \to 0$ pt.-wise but $\int f_n = 1$, $\int f = 0$.

e.g.

$$f_n(x) = n \cdot 1_{[n,n+1]}(x)$$

$$f_n(x) \to \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

So $f_n(x)$ caveges a.e.

$\int f_n = 1$ but $\int f = 0$

e.g.

$$f_n(x) = \frac{1}{n} 1_{[n,n+1]}(x)$$

$$f_n(x) \to 0$$