7.2.1. Prove Lemma 7.2.5:

(v) No box is needed to cover $\emptyset$, so $m^*(\emptyset) = 0$.

(vi) $\inf\{\sum_{i=1}^{\infty} vol(B_i)\} \geq 0$ because $vol(B_i) \geq 0 \ \forall B_i$. Thus, positivity holds.

(vii) Any open cover of B is also an open cover of A. Therefore,
$\inf\{\sum_j vol(B_j) : (B_j)_j \text{ covers } A\} \leq \inf\{\sum_j vol(B_j) : (B_j)_j \text{ covers } B\}$, so
$m^*(A) \leq m^*(B)$ if $A \subseteq B$.

(viii) $\forall \varepsilon > 0$ such that for any $A_j, \exists \{B_j\}$ s.t. $\sum |B_j| \leq m^*(A_j) + \varepsilon/n$
Taking the union of all such $\{B_j\}$ gives an open cover of $\bigcup_{j \in J} A_j$, with total volume $\leq \sum m^*(A_j) + \varepsilon$
Thus, $m^*(\bigcup_{j \in J} A_j) \leq \sum_{j \in J} m^*(A_j)$

(x) Take a similar approach to (viii), but choose $\{B_j\}$ such that $\sum |B_j| \leq m^*(A_j) + \varepsilon/2^j$. Taking the union of all $\{B_j\}$ gives an open cover of $\bigcup_{j \in J} A_j$, with total volume $\leq \sum m^*(A_j) + \varepsilon \quad \forall \varepsilon > 0$.
Thus, $m^*(\bigcup_{j \in J} A_j) \leq \sum_{j \in J} m^*(A_j)$.

(xiii) Consider any $\Omega \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$. For any open cover of $\Omega$, an open cover of $\Omega + x$ of equal volume can be obtained by shifting each box by $x$, and vice versa by shifting each box by $-x$. Thus, the sets of volumes of covers for $\Omega$ and $\Omega + x$ are identical,
so $m^*(\Omega + x) = m^*(\Omega)$.

7.2.2. $\forall \varepsilon > 0$, $\exists \{A_j\}$ s.t. $\sum vol(A_j) \le m_n^*(A) + \varepsilon$ and $\{A_j\}$ covers $A$, and $\exists \{B_k\}$ s.t. $\sum vol(B_k) \le m_m^*(B) + \varepsilon$, $\{B_k\}$ covering $B$.
Consider the cover $\{A_j\} \times \{B_k\}$ of $A \times B$, where $\{A_j\} \times \{B_k\}$ consists of an "extended box" for each pair $A_j \in \{A_j\}$, $B_k \in \{B_k\}$.
$A_j \subset \mathbb{R}^n = \prod_{i=1}^{n} (a_i, b_i)$, with $B_k = \prod_{j=1}^{m} (a_j', b_j')$ becomes $\prod_{i=1}^{n} (a_i, b_i) \prod_{j=1}^{m} (a_j', b_j') \subset \mathbb{R}^{n+m}$.
Note that the volume of all the boxes in $\{A_j\} \times \{B_k\}$ is equal to $(\sum vol(A_j))(\sum vol(B_k)) \le (m_n^*(A) + \varepsilon)(m_m^*(B) + \varepsilon)$.
Taking $\varepsilon$ to $0$ gives $m^*(A \times B) \le m^*(A) m^*(B)$.

7.2.3. (a) Revised solution using Discord discussion:
$\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} (A_j \setminus \bigcup_{i<j} A_i)$, disjointizing the sets
$m(\bigcup A_j) = m(\bigcup (A_j \setminus \bigcup_{i<j} A_i)) = \sum_{j=1}^{\infty} m(A_j \setminus \bigcup_{i<j} A_i) = \lim_{j \to \infty} m(A_j)$

(b) $A_j = \bigcap_{j=1}^{\infty} A_j \cup \bigcup_{k=j}^{\infty} (A_k \setminus A_{k+1})$, a disjoint union
$m(A_j) = m(\bigcap_{j=1}^{\infty} A_j) + \sum_{k=j}^{\infty} m(A_k \setminus A_{k+1})$
$\lim_{j \to \infty} m(A_j) = m(\bigcap_{j=1}^{\infty} A_j) + \lim_{j \to \infty} \sum_{k=j}^{\infty} m(A_k \setminus A_{k+1}) = m(\bigcap_{j=1}^{\infty} A_j)$

7.2.4. For every $q \geq 1$, $(0,1)^n$ can contain $q^n$ disjoint translates of $(0, 1/q)^n$. By normalization, $m((0,1)^n) = 1$, so $q^n m((0, 1/q)^n) \leq 1$, Thus, $m((0, 1/q)^n) \leq q^{-n}$.

$(0,1)^n$ can be covered by $q^n$ disjoint translates of $[0, 1/q]^n$, so $m((0,1)^n) \leq q^n m([0, 1/q]^n)$, so $m([0, 1/q]^n) \geq q^{-n}$.

Now, we will show $m([0, 1/q]^n \setminus (0, 1/q)^n) \leq \varepsilon$, because the boundary of $[0, 1/q]^n$ can be covered by boxes with $\varepsilon$ as a dimension across the boundary, $\varepsilon$ arbitrarily small. Therefore, $m((0, 1/q)^n) + \varepsilon = m([0, 1/q]^n)$ by finite additivity, so we can conclude

$$m((0, 1/q)^n) = m([0, 1/q]^n) =$$

7.4.1. Let $A_+ = A \cap (0,\infty)$ and $A_- = A \setminus (0,\infty)$. $A = A_+ \cup A_-$, so by sub-additivity, $m^*(A) \le m^*(A_+) + m^*(A_-)$

Consider an open cover of A of open boxes $\{B_j\}$, with $\sum_{j=1}^{\infty} |B_j| \le m^*(A) + \frac{\varepsilon}{2}$.

Let $B_j^+ = B_j \cap (0,\infty)$, $B_j^- = B_j \cap (-\infty, \frac{\varepsilon}{2^{j+1}})$. $B_j = B_j^- \cup B_j^+$, and

$|B_j| + \frac{\varepsilon}{2^{j+1}} \ge |B_j^+| + |B_j^-| \ge |B_j|$

$A_+ \subset \bigcup B_j^+$, $A_- \subset \bigcup B_j^-$, so $m^*(A_+) + m^*(A_-) \le \sum |B_j^+| + \sum |B_j^-| \le \sum_{j=1}^{\infty} (|B_j| + \frac{\varepsilon}{2^{j+1}})$

$\le (\sum |B_j|) + \frac{\varepsilon}{2} \le m^*(A) + \varepsilon$, so $m^*(A_+) + m^*(A_-) \le m^*(A)$.

Thus, $m^*(A) = m^*(A_+) + m^*(A_-)$.

7.4.2. Since A is an open box, let A be expressed as

$$\prod_{i=1}^{n} (a_i, b_i) = \left(\prod_{i=1}^{n-1} (a_i, b_i)\right)(a_n, b_n)$$

Letting $A_n = (a_n, b_n)$ and using 7.4.1,

$m^*(A_n) = m^*(A_n \cap (0,\infty)) + m^*(A_n \setminus (0,\infty))$, and

$$m^*(A) = m^*\left(\prod_{i=1}^{n-1} (a_i, b_i)\right)\left(m^*(A_n \cap (0,\infty)) + m^*(A_n \setminus (0,\infty))\right)$$

$$= m^*(A \cap E) + m^*(A \setminus E)$$

7.4.3. By finite sub-additivity, since $A = (A \cap E) \cup (A \setminus E)$, we have $m^*(A) \leq m^*(A \cap E) + m^*(A \setminus E)$, for any $A \subseteq \mathbb{R}^n$

For $\varepsilon > 0$, we can find an open cover of A of boxes $\{B_k\}$ such that $\sum_k |B_k| \leq m^*(A) + \varepsilon$.

$A \cap E \subseteq \bigcup_k (B_k \cap E)$, so $m^*(A \cap E) \leq m^*\left(\bigcup_k (B_k \cap E)\right) \leq \sum_k m^*(B_k \cap E)$.

Similarly, $m^*(A \setminus E) \leq \sum_k m^*(B_k \setminus E)$.

Then, $m^*(A \cap E) + m^*(A \setminus E) \leq \sum_k m^*(B_k \cap E) + m^*(B_k \setminus E)$, which by 7.4.2, equals $\sum_k m^*(B_k) = \sum |B_k| \leq m^*(A) + \varepsilon$.

Thus, $m^*(A \cap E) + m^*(A \setminus E) \leq m^*(A)$.

Therefore, $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$, so half-spaces are measurable.

7.4.4.

(a) If $\forall A \subseteq \mathbb{R}^n$, $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$, then $m^*(A) = m^*(A \setminus (\mathbb{R}^n \setminus E)) + m^*(A \cap (\mathbb{R}^n \setminus E))$, so $\mathbb{R}^n \setminus E$ is measurable,

(b) Outer measure is translation invariant. Thus, $\forall A \subseteq \mathbb{R}^n$, we have

$$m^*(A) = m^*(-x + A) = m^*((-x+A) \cap E) + m^*((-x+A) \setminus E)$$
$$= m^*(A \cap E+x) + m^*(A \setminus E+x).$$

Thus, $E+x$ is measurable, and $m(E) = m(x+E)$ because $m^*(E) = m^*(x+E)$

(c) For some $A \subseteq \mathbb{R}^n$, define $A_{++} = A \cap E_1 \cap E_2$, $A_{+-} = A \cap E_1 \cap E_2^c$, $A_{-+} = A \cap E_1^c \cap E_2$, $A_{--} = A \cap E_1^c \cap E_2^c$

Since $E_1$ is measurable, $m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c)$
$= m^*(A_{+-} \cup A_{++}) + m^*(A_{-+} \cup A_{--})$.

Since $E_2$ is measurable, $m^*(A_{+-}) + m^*(A_{++}) = m^*(A_{+-} \cup A_{++})$ and $m^*(A_{--}) + m^*(A_{-+}) = m^*(A_{-+} \cup A_{--})$.

Thus, $m^*(A) = m^*(A_{++}) + m^*(A_{+-}) + m^*(A_{-+}) + m^*(A_{--})$.

Doing this same split on $A_{+-} \cup A_{-+} \cup A_{--}$ yields

$$m^*(A) = m^*(A_{++}) + m^*(A_{+-} \cup A_{-+} \cup A_{--})$$
$$= m^*(A \cap (E_1 \cap E_2)) + m^*(A \setminus (E_1 \cap E_2)),$$ so $E_1 \cap E_2$ is measurable.

Showing measurability of $E_1 \cup E_2$ is similar:

$$m^*(A) = m^*(A_{--}) + m^*(A_{++} \cup A_{+-} \cup A_{-+})$$
$$= m^*(A \setminus (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)).$$

(d) Measurability of $\bigcup_{j=1}^N E_j$ and $\bigcap_{j=1}^N E_j$ can be shown inductively by applying part (c) during the inductive step.

(e) The open box $\prod_{i=1}^n (a_i, b_i)$ is the same as $\bigcap_{i=1}^n \{(x_1, \dots, x_n) : x_i > a_i\} \cap \{(x_1, \dots, x_n) : x_i < b_i\}$

The closed box $\prod_{i=1}^n [a_i, b_i]$ is the complement of $\bigcap_{i=1}^n \{(x_1, \dots, x_n) : x_i < a_i\} \cap \{(x_1, \dots, x_n) : x_i > b_i\}$. Both are measurable by b), d), and the lemma that half-spaces are measurable.

7.4.4.

(F) $A \cap E \subseteq E$, so $m^*(A \cap E) \leq 0$. By positivity, $m^*(A \cap E) = 0$.

$A \setminus E \subseteq A$, so $m^*(A \setminus E) \leq m^*(A)$.

By finite sub-additivity, $m^*(A \cap E) + m^*(A \setminus E) \geq m^*(A)$.

Thus, $m^*(A) = m^*(A \setminus E) + m^*(A \cap E)$, so $E$ is measurable.