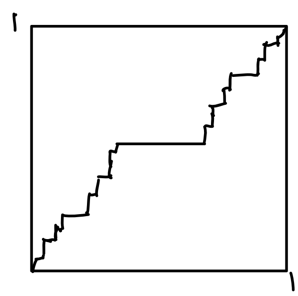
I will discuss three interesting counter-examples related to the course content covered in Math 105 this semester.

1. Continuous function with derivative f almost everywhere, but does not differ from the indefinite integral of f by a constant.

We define a function $G: [0,1] \to [0,1]$ as follows:
For any $x \in [0,1]$, express x in base 3, and then truncate the number after the first 1. $G(x)$ is the result of interpreting this truncated number as a base 2 number.
G is called the Cantor function, or the Devil's Staircase.
Let C be the Cantor set, defined by $C_0 = [0,1]$, $C_n = \frac{C_{n-1}}{3} \cup \frac{2+C_{n-1}}{3}$, and $C = \lim_{n\to\infty} C_n$.
Then, G is constant on each interval of $[0,1] \setminus C$.
Below, I've included an approximate drawing of the Cantor function. Notice that it is constant on each interval which is removed when constructing the Cantor set, and only increases on points in the Cantor set.



Since we know the Cantor set has measure zero, the derivative of G is zero almost everywhere.
However, it is clear that the indefinite integral of zero, which is still zero, does not just differ from G by a constant, since G is not constant.

This counter-example reveals the importance of introducing absolutely continuous functions:

<u>Def</u>: $G: [a,b] \to \mathbb{R}$ is absolutely continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. whenever $I_1, \dots, I_n$ are disjoint intervals in $[a,b]$,

$$\sum_{i=1}^{n} (b_i - a_i) < \delta \implies \sum_{i=1}^{n} \left| G(b_i) - G(a_i) \right| < \varepsilon$$

This allows us to state Lebesgue's Antiderivative Theorem - one direction of the theorem states that if a function is absolutely continuous, it is the indefinite integral of its derivative, up to a constant.

F_σ-sets and G_δ-sets are important, a primary reason being that all F_σ-sets and G_δ-sets are measurable.

Below, we look at a set that is not F_σ and a set that is not G_δ.

2. A set that is not a F_σ-set.

Recall that a F_σ-set is a set which is a countable union of closed sets.

The set of irrational numbers is not a F_σ-set. Suppose for the sake of contradiction that $\mathbb{R} \setminus \mathbb{Q} = C_1 \cup C_2 \cup \dots$, for C_i closed.

First, a few definitions:

<u>Def</u>: Interior point, a point contained in the union of all open subsets.

<u>Def</u>: Nowhere dense, a set whose closure has no interior points.

<u>Def</u>: A metric space (X, d) is <u>complete</u> if every Cauchy sequence in X converges in X

Baire Category Theorem : A complete metric space can not be written as a countable union of nowhere dense closed subsets.

Since subsets of $\mathbb{R}\setminus\mathbb{Q}$ have no interior points, $C_1 \cup \cdots$ is a countable union of nowhere dense sets.

Note that $\mathbb{Q}$ can be written as $\bigcup_{q\in\mathbb{Q}} \{q\}$, so this representation of $\mathbb{R}\setminus\mathbb{Q}$ gives $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R}\setminus\mathbb{Q})$ in terms of a countable union of nowhere dense closed subsets, which contradicts Baire's Category Theorem.

Thus, $\mathbb{R}\setminus\mathbb{Q}$ can not be a F_σ- set.

3. A set that is not a G_δ-set.

The complement of a F_σ-set is a G_δ-set (by applying De Morgan's Laws). The converse is also true: the complement of a G_δ-set is a F_σ-set.

Since we previously showed that $\mathbb{R}\setminus\mathbb{Q}$ is not a F_σ-set, we can conclude that $\mathbb{R}\setminus(\mathbb{R}\setminus\mathbb{Q}) = \mathbb{Q}$ is not a G_δ-set.