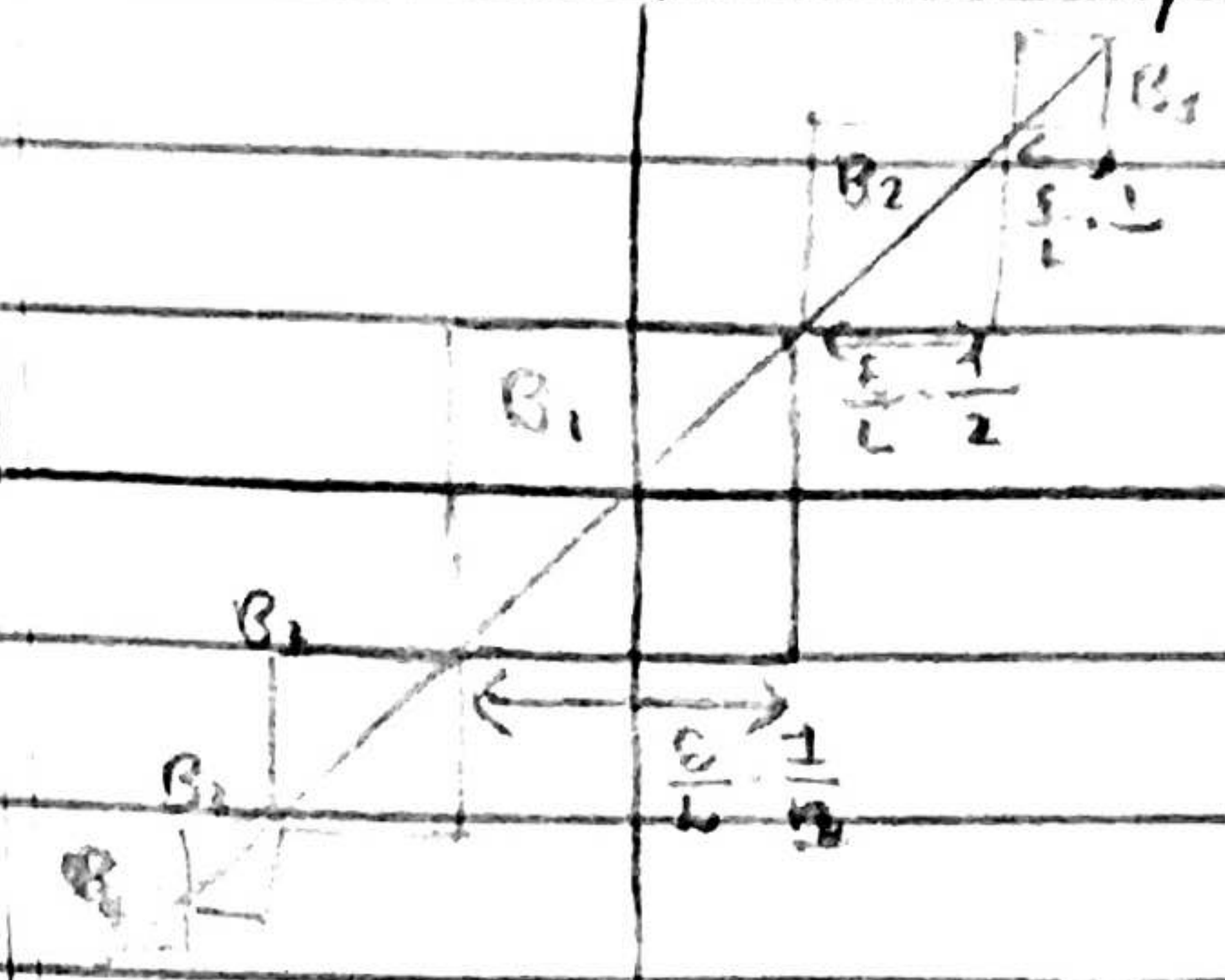
1) First consider an open covering of $\{y=x\} \subseteq \mathbb{R}^2$



Let $\varepsilon > 0$. Let $\sum_{n=1}^{\infty} \frac{1}{n^2} = L$
(where L is finite because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges)
~~Let B_i be the box with~~

Let B_n be the $\frac{\sqrt{\varepsilon}}{\sqrt{L}n} \times \frac{\sqrt{\varepsilon}}{\sqrt{L}n}$ box, with area $\frac{\varepsilon}{Ln^2}$
(where ε and L are constants)
Let (B_i) be the collection of boxes B_n covering $\{x=y\}$ as shown.

$\sum_{n=1}^{\infty} \frac{\sqrt{\varepsilon}}{\sqrt{L}n}\sqrt{2} = \frac{\sqrt{\varepsilon}\sqrt{2}}{\sqrt{L}} \sum_{n=1}^{\infty} \frac{1}{n}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\Rightarrow (B_i)$ covers $\{x=y\}$ on the diagonal.

$\cancel{m(B}\ m(\{x=y\}) \leq \cancel{Vol}\ \sum Vol(B_i) = \sum_{n=1}^{\infty} \frac{\sqrt{\varepsilon}^2}{Ln^2} = \frac{\varepsilon}{L} \sum_{n=1}^{\infty} \frac{1}{n^2} = \varepsilon$

$\therefore m(\{x=y\}) \leq \varepsilon \quad \forall \varepsilon > 0$

$\Rightarrow m(\{x=y\}) = 0$

Next consider the set $E = \{x=0\}$ under the affine linear transformation $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\{x=0\}A = \{y=x\}$

$\det(A) = 1$

$m(\{y=x\}) = m(EA) = \det(A)m(E) = m(E) = m(\{x=0\})$

$\{x=0\}$ has measure 0 in $\mathbb{R}^2$

$\therefore m(\{y=x\}) = 0$

Finally, consider the slice E_x $\quad E = \{y=x\}, \ x \in \mathbb{R}$

$E_x = \{x\} \subseteq (x + \frac{\varepsilon}{2}, x - \frac{\varepsilon}{2}) \quad \forall \varepsilon > 0$

$m(E_x) \leq m(x + \varepsilon/2, x - \frac{\varepsilon}{2}) = \varepsilon$

$\therefore m(E_x) = 0 \quad \forall x \in \mathbb{R}$

By the zero-slice theorem, given that every slice of E is a null set, E is a null set

More generally, any straight line in $\mathbb{R}^n$ can be written as an affine transformation of a co-ordinate axis, which will always have measure zero.
Likewise, any ~~plate~~ hyper plane in $\mathbb{R}^n$ of dimension $n-1$ or less can be written as an affine transformation of a co-ordinate plane which will have measure zero.

2) Lemma: 16: Every open set in n-space is a countable disjoint union of open cubes plus a zero set
Let $U \subset \mathbb{R}^n$ be an open set.
Partition $\mathbb{R}^n$ into disjoint n-dimensional unit cubes. Accept every cube which is contained in U. Reject every cube which is disjoint with U. Partition every cube which partially intersects U into 2^n smaller cubes, and repeat the process ad infinitum. ~~Fe~~ Let B be the union of these disjoint dyadic cubes. Let Z be the set of boundary points. Each boundary is a horizontal or vertical $n-1$ dimensional hyper-plane, so will have measure zero in $\mathbb{R}^n$, and there are countably many of them

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* note: I proved the wrong lemma 16 and I'm still working on the other proof.

Let $U \subset R^n$ be an open set.
Partition R^n into disjoint n-dimensional unit cubes. Accept every cube which is contained in U. Reject every cube which is disjoint with U. Partition every cube which partially intersects U into 2^n smaller cubes, and repeat the process ad infinitum. ~~Fe~~ Let B be the union of these disjoint dyadic cubes. Let Z be the set of boundary points. Each boundary is a horizontal or vertical $n-1$ dimensional hyper-plane, so will have measure zero in R^n, and there are countably many of them

$U \subset B \cup Z$, where B is a countable union of disjoint cubes, and Z is a zero set

Lemma 21:

Let $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ be measurable

Let's start by generalising lemmas 23 & 24

Suppose $m(A)=0$.

$\Rightarrow \forall \varepsilon$ we can cover A with open boxes ~~with~~ I_i ~~with~~ with volume $Vol(I_i) < \frac{\varepsilon}{n^m 2^{n+1}}$ $\forall n \in \mathbb{N}$

Let $B_n = (I_i) \times [-\frac{n}{2}, \frac{n}{2}]$

$Vol(B_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{n^m 2^{n+1}} \cdot n^m = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \varepsilon$

$B \times A \subset B_n \Rightarrow m(B \times A) \leq \varepsilon \quad \forall \varepsilon$

$\Rightarrow m(A \times B) = 0 \qquad$ (Lemma 23)

Now suppose A and B are open

By Lemma 16, $A = \bigcup_i I_i \cup Z_A$, $B = \bigcup_j J_j \cup Z_B$

where I_i, ~~B~~ J_j are disjoint open sets, Z_A, Z_B are zero sets.

So $A \times B = \bigcup_{i,j} I_i \times J_j \cup (Z_A \times B) \cup (Z_B \times A)$

By Lemma 23, $(Z_A \times B), (Z_B \times A)$ are zero sets

So $A \times B = \bigcup_{i,j} I_i \times J_i$

$\Rightarrow m(A \times B) = m(\bigcup_{i,j} I_i \times J_i) = \sum_{i,j} Vol(I_i \times J_j) = \sum_{i,j} Vol(I_i) Vol(J_j)$

$= \sum_i Vol\, I_i \sum_j Vol\, J_j = m(A)\, m(B) \qquad$ (Lemma 24)

Let K_A, K_B be kernels of A & B respectively, and H_A, H_B be hulls. We have that

$m(K_A \times K_B) = m(A \times B) = m(H_A \times H_B)$

So $A \times B$ is measurable.

Now by partitioning A ~~in~~ and B into countable unions of bounded sets $A \times [-k,k]^n$, $B \times [-k,k]^m$ $\forall k \in \mathbb{N}$

we can use measure continuity to show
$m(A)m(B) = m(A \times B)$

3) First lets prove $J^*A = J^*\bar{A}$
$A \subset \bar{A} \Rightarrow J^*A \leq J^*\bar{A}$
Let $\varepsilon > 0$ Let (I_k) be a finite covering of A, where I_k is an open interval $\forall k \leq n$
If $I_k = (a_k, b_k)$, let $J_k = (a_k - \frac{\varepsilon}{3n}, b_k + \frac{\varepsilon}{3n})$
$\bar{A} \subset (J_k)$ ~~and~~ $\cancel{\sum^* J_k}$ $\forall \sum_{k=1}^{n} |J_k| < \sum_{k=1}^{n} |I_k| + \varepsilon$
$\therefore \forall I_k, \exists J_k$ s.t $\sum_{k=1}^{n} |J_k| < \sum_{k=1}^{n} |I_k| + \varepsilon$
$\Rightarrow \inf(\sum_{k=1}^{n} |J_k|) \leq \inf(\sum_{n=1}^{n} |I_k|)$
$\Rightarrow \cancel{J^*(A)}$ $J^*\bar{A} \leq J^*A$
So $J^*A = J^*\bar{A}$
Now proving $J^*\bar{A} = m\bar{A}$
$m\bar{A} = \inf\{\sum |B_i| \mid B_i \text{ is a countable covering of } \bar{A}\} \leq$
$\inf\{\sum |I_k| \mid I_k \text{ is a finite covering of } \bar{A}\} = J^*\bar{A}$
Let (B_i) be a countable covering of $\bar{A}$.
~~$\bar{A}$ is closed. Let $\bar{A}_n = \bar{A} \cap [-n, n]$ $\forall n \in \mathbb{N}$~~
~~$\bar{A}_n$ is closed and bounded~~
$\bar{A}$ is closed and bounded $\Rightarrow \bar{A}$ is compact
$\Rightarrow$ for every open cover of $\bar{A}$ $\exists$ a finite subcover.
$\therefore$ if B_i is an open cover, $\exists I_k$ which is finite s.t
$\bar{A} \subset (I_k) \subset (B_i) \Rightarrow \inf\{|I_k| \mid I_k \text{ covers } \bar{A}\} \leq \inf\{|B_i| \mid B_i \text{ covers } \bar{A}\}$
$\Rightarrow J^*\bar{A} \leq m\bar{A}$
$\therefore J^*\bar{A} = m\bar{A}$