1) Let $p > 2$. Let $c > 0$

Let $E = \{x \in [0,1] \mid |x - \frac{a}{q}| < \frac{c}{q^p}$ for infinitely many positive integers a and $q\}$

By the archimedian principle $\exists N \in \mathbb{N}$ s.t $0 < 2c < N$

$\Rightarrow 0 < \frac{2c}{N} < 1$

If $N \leq q < a$ then $0 < \frac{\cancel{2}c}{a} < \frac{c}{q} < \frac{c}{N} < \frac{2c}{N} < 1$

$x - \frac{a}{q} \leq x - \frac{q+1}{q} \leq 1 - \frac{q+1}{q} = -\frac{1}{q}$ (since $x \in [0,1]$)

$\Rightarrow |x - \frac{a}{q}| \geq \frac{1}{q} \geq \frac{1}{q^{p-1}} \geq \frac{1}{q^{p-1}} \cdot \frac{\cancel{2}c}{N} \geq \frac{\cancel{2}c}{q^p}$

$\therefore$ for $N \leq q < a$, ~~$m(E)$~~ $= E = \emptyset$ so $m(E) = 0$

$\Rightarrow$ for $q < a$, only finitely many positive integers a, q satisfy $x \in [0,1]$, $|x - \frac{a}{q}| < \frac{c}{q^p}$

$\Rightarrow$ for members of E we can consider only positive integers a and q s.t $a < q$

Let $\Omega_q = \{x \in [0,1] \mid |x - \frac{a}{q}| < \frac{c}{q^p}$ for some $a < q\}$

$x \in \Omega_q \Leftrightarrow \frac{a}{q} - \frac{c}{q^p} \leq x \leq \frac{a}{q} + \frac{c}{q^p}$

$\therefore \Omega_q = [\frac{1}{q} - \frac{c}{q^p}, \frac{a}{q} + \frac{c}{q^p}] \cup \dots \cup [\frac{a}{q} - \frac{c}{q^p}, \frac{q}{q} + \frac{c}{q^p}]$

$\Omega_q \subset [0,1] \ \forall q \Rightarrow m(\Omega_q) \leq 1$

$\forall q > N$, $\frac{1}{q} \geq \frac{2c}{q^p} \Rightarrow$ the intervals $[\frac{a}{q} - \frac{c}{q^p}, \frac{a}{q} + \frac{c}{q^p}]$ are disjoint

$\Rightarrow \forall q > N$, $m(\Omega_q) = \frac{2c}{q^p}(q-1) = \frac{2c}{q^{p-1}} - \frac{2c}{q^p}$

$\therefore \sum_{q=1}^{\infty} m(\Omega_q) = \sum_{q=1}^{N} m(\Omega_q) + \sum_{q=N}^{\infty} m(\Omega_q)$

$\leq N + \sum_{q=N}^{\infty} \frac{2c}{q^{p-1}} - \frac{2c}{q^p}$

since $p > 2$, $p-1$ and $p > 1$

$\Rightarrow \sum_{q=N}^{\infty} \frac{2c}{q^{p-1}} - \frac{2c}{q^p}$ is finite.

$\therefore \sum m(\Omega_q)$ is finite

$E = \{x \in \Omega_q$ for infinitely many $q\}$

$\therefore$ by the Borel-Cantelli lemma, $m(E) = 0$

2) Let $f_n : \mathbb{R} \to [0,\infty)$ be a sequence of non-negative measurable functions s.t $\int_{\mathbb{R}} f_n \leq \frac{1}{4^n}$

Let $S_n = \{x \in \mathbb{R} \mid f_n(x) > \frac{1}{\varepsilon 2^{n+1}}\}$, $\varepsilon > 0$, $n \in \mathbb{N}$

$\{f_n(x) > \frac{1}{\varepsilon 2^{n+1}}\}$ is open in $\mathbb{R}$ and f is measurable

$\Rightarrow S_n$ is measurable

Let $g : \mathbb{R} \to [0,\infty)$ s.t $g(x) = \begin{cases} \frac{1}{\varepsilon 2^{n+1}} & x \in S_n \\ 0 & \text{otherwise} \end{cases}$

Let $h : \mathbb{R} \to [0,\infty)$ s.t $h(x) = \begin{cases} 0 & x \in S \\ f(x) & \text{otherwise} \end{cases}$

g and h are measurable functions

~~&~~ g is a simple functions

$$\Rightarrow \int_{\mathbb{R}} g = 0 \cdot m(S_n^c) + \frac{1}{\varepsilon 2^{n+1}} m(S_n)$$
$$= \frac{1}{\varepsilon 2^{n+1}} m(S_n)$$

$g(x) + h(x) \leq f(x) \ \forall x \in \mathbb{R}$

$$\Rightarrow \int_{\mathbb{R}} g + h \leq \int_{\mathbb{R}} f \leq \frac{1}{4^n}$$

$$\Rightarrow \frac{1}{\varepsilon 2^{n+1}} m(S_n) \leq \frac{1}{4^n} = \frac{1}{2^{2n}}$$

$$\Rightarrow m(S_n) \leq \frac{\varepsilon 2^n}{2^{2n}} = \frac{\varepsilon}{2^n}$$

Let $\cancel{S=} \bigcup_{n \in \mathbb{N}} S_n = E$ ~~Let $\delta > 0$~~

~~$\forall x \in \mathbb{R} \setminus E$~~

Let $\delta > 0$ Take $N \in \mathbb{N}$, $N > \log_2\left(\frac{1}{\varepsilon\delta}\right)$

$\forall x \in \mathbb{R} \setminus E$, $n > N$, $f_n(x) \leq \frac{1}{\varepsilon 2^n} \leq \frac{1}{\varepsilon 2^N} = \delta$

$\therefore$ for $x \in \mathbb{R} \setminus E$, $f_n \to 0$ pointwise

$$m(E) \leq \sum_{n \in \mathbb{N}} m(S_n) = \sum_{n \in \mathbb{N}} \frac{\varepsilon}{2^n} = 2\varepsilon$$

$\therefore \forall \varepsilon$ we can find ~~m~~ E s.t $f_n \to 0$ pointwise $\forall x \in \mathbb{R} \setminus E$ and $m(E) \leq \varepsilon$.

3) Let $f_n : [0,1] \to [0,\infty)$ be a sequence of non-negative measurable functions s.t $f_n \to 0$ pointwise

Let $\varepsilon > 0$. Let $S_N = \{x \in [0,1] \mid f_n(x) \leq \varepsilon \ \forall n > N\}$

$\therefore$ ~~$S_1 \supset S_2 \supset S_3 \supset \ldots S_1 \subset S_2$~~ $S_1 \subset S_2 \subset S_3 \ldots$

$\Rightarrow m(S_1) \leq m(S_2) \leq m(S_3) \leq \ldots$

Note that $f_n \to 0$ pointwise

$\Rightarrow \forall x \ \exists N$ s.t $\forall n > N$, $f_n(x) \leq \varepsilon$

$\Rightarrow \lim_{n \to \infty} S_n = [0,1]$

$\therefore$ by measure continuity, $\lim_{n\to\infty} m(S_n) = m[0,1] = 1$

$\Rightarrow \exists M \in \mathbb{N}$ s.t $|m(S_M) - 1| < \varepsilon$

$\Rightarrow 1 - \varepsilon < m(S_M)$

~~Let~~ $S_M^c = \{x \in [0,1] \mid f_n(x) > \varepsilon \ \forall n > M\}$

$m(S_M) + m(S_M^c) = 1$

$\Rightarrow 1 - \varepsilon < 1 - m(S_M^c) \Rightarrow m(S_M^c) < \varepsilon$

$\forall x \in [0,1] \setminus S_M^c$, $n > M \Rightarrow f_n(x) \leq \varepsilon$

~~and ε $m(S_M^c)$~~

$\therefore$ for $x \in [0,1] \setminus S_M^c$, $f_n(x) \to 0$ uniformly

and $m(S_M^c) < \varepsilon \quad \forall \varepsilon > 0$