1)a) $\int_\Omega |f| < \infty \quad \int_\Omega |g| < \infty \quad |c| < \infty$

$\int_\Omega |cf| = \int_\Omega |c||f| = |c| \int_\Omega |f| < \infty$

$\therefore$ cf is absolutely integrable

$\int_\Omega cf = \int_\Omega (cf)^+ - \int_\Omega (cf)^-$

$c = 0 \implies \int_\Omega cf = \int_\Omega 0 = 0 = 0 \int_\Omega f = c \int_\Omega f$

$c > 0 \implies (cf)^+ = c(f^+), \ (cf)^- = c(f^-)$

$\implies \int_\Omega cf = \int_\Omega (cf)^+ - \int_\Omega (cf)^- = \int_\Omega cf^+ - \int_\Omega cf^-$

$= c \int_\Omega f^+ - c \int_\Omega f^- = c\left(\int_\Omega f^+ - \int_\Omega f^-\right) = c \int_\Omega f$

$c < 0 \implies (cf)^+ = -cf^- \quad (cf)^- = -cf^+$

$\implies \int_\Omega cf = \int_\Omega (cf)^+ - \int_\Omega (cf)^- = \int_\Omega -cf^- - \int_\Omega -cf^+$

$= -c \int_\Omega f^- - -c \int_\Omega f^+ = c\left(\int_\Omega f^+ - \int_\Omega f^-\right) = c \int_\Omega f$

$\therefore \forall c \in \mathbb{R}, \ \int_\Omega cf = c \int_\Omega f$

b) $\int_\Omega |f| < \infty \quad \int_\Omega |g| < \infty$

$\int_\Omega |f+g| \leq \int_\Omega |f| + |g| = \int_\Omega |f| + \int_\Omega |g| < \infty$

$\therefore$ $f+g$ is absolutely integrable

Define the following subsets of Ω:

$A^+ = \{x \in \Omega \mid f(x), g(x) \geq 0\}$ $A^- = \{x \in \Omega \mid f(x), g(x) < 0\}$

$B^+ = \{x \in \Omega \mid f(x) > 0, \ g(x) < 0, \ |f(x)| > |g(x)|\}$

$B^- = \{x \in \Omega \mid f(x) < 0, \ g(x) > 0, \ |f(x)| > |g(x)|\}$

$C^+ = \{x \in \Omega \mid f(x) < 0 \ g(x) > 0 \ |f(x)| < |g(x)|\}$

$C^- = \{x \in \Omega \mid f(x) > 0 \ g(x) < 0 \ |f(x)| < |g(x)|\}$

$x \in A^+ \implies (f+g)^+(x) = f^+(x) + g^+(x)$

$(f+g)^-(x) = f^-(x) = g^-(x) = 0$

$x \in B^+ \implies (f+g)^+(x) = f^+(x) - g^-(x)$

$(f+g)^-(x) = f^-(x) = g^+(x) = 0$

$x \in C^+ \implies (f+g)^+(x) = g^+(x) - f^-(x)$

$x \in A^- \Rightarrow (f+g)^+(x) = f^+(x) = g^+(x) = 0$

$\qquad (f+g)^-(x) = f^-(x) + g^-(x) = 0$

$x \in B^- \Rightarrow (f+g)^+(x) = f^+(x) = g^-(x) = 0$

$\qquad (f+g)^-(x) = -(f^-(x) - g^+(x)) = g^+(x) - f^-(x)$

$x \in C^- \Rightarrow (f+g)^+(x) = g^+(x) = f^-(x) = 0$

$\qquad (f+g)^-(x) = -(g^-(x) - f^+(x)) = f^+(x) - g^-(x)$

$\cancel{\therefore \forall}\ \Omega = A^+ \cup A^- \cup B^+ \cup B^- \cup C^+ \cup C^-$

$\therefore \forall x \in \Omega,\ (f+g)^+(x) - (f+g)^-(x) = f^+(x) + g^+(x) - f^-(x) - g^-(x)$

$\Rightarrow \int_\Omega (f+g)^+ \cancel{+} - \int_\Omega (f-g)^- = \int_\Omega f^+ - \int_\Omega f^- + \int_\Omega g^+ - \int_\Omega g^-$

$\Rightarrow \int_\Omega (f+g) = \int_\Omega f + \int_\Omega g$ as required

c) $\int_\Omega f = \int_\Omega f^+ - \int_\Omega f^- \quad \int_\Omega g = \int_\Omega g^+ - \int_\Omega g^-$

$f(x) \leq g(x) \Rightarrow f^+(x) \leq g^+(x),\ g^-(x) \leq f^-(x)$

$\Rightarrow \int_\Omega f^+(x) \leq \int_\Omega g^+(x),\ \int_\Omega g^-(x) \leq \int_\Omega f^-(x)$

$\Rightarrow \int_\Omega f^+ - \int_\Omega f^- \leq \int_\Omega g^+ - \int_\Omega g^-$

$\Rightarrow \int_\Omega f \leq \int_\Omega g$

d) $f(x) = g(x)$ for almost every $x \in \Omega$

Let $f = h + F \quad g = h + G$ where

$h = \begin{cases} f & f = g \\ 0 & \text{otherwise} \end{cases} \quad F = \begin{cases} f & f \neq g \\ 0 & \text{otherwise} \end{cases} \quad G = \begin{cases} g & f \neq g \\ 0 & \text{otherwise} \end{cases}$

$\int_\Omega f = \int_\Omega h + F = \int_\Omega h + \int_\Omega F$

$\int_\Omega g = \int_\Omega h + G = \int_\Omega h + \int_\Omega G$

Note that F and G are zero almost everywhere

$\Rightarrow \int_\Omega F = \int_\Omega G = 0$

$\Rightarrow \int_\Omega f = \int_\Omega h + 0 = \int_\Omega h = \int_\Omega \cancel{G}\, g$

$\therefore \int_\Omega f = \int_\Omega g$

2) $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) \leq g(x)$ $\forall x \in \mathbb{R}$, $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$

Let $S = \{x \in \mathbb{R} \mid f(x) \neq g(x)\}$

$\int_{\mathbb{R}} f = \int_{\mathbb{R} \setminus S} f + \int_S f$ $\int_{\mathbb{R}} g = \int_{\mathbb{R} \setminus S} g + \int_S g$

$\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$ ~~and~~ $\cancel{\int_{\mathbb{R} \setminus S}}$

$f(x) = g(x)$ $\forall x \in \mathbb{R} \setminus S \Rightarrow \int_{\mathbb{R} \setminus S} f = \int_{\mathbb{R} \setminus S} g$

$\therefore \int_S f = \int_S g$

Note that by definition, $f(x) \neq g(x)$ $\forall x \in S$

$\Rightarrow f(x) < g(x)$ $\forall x \in S$

$\Rightarrow \exists \varepsilon > 0$ $\forall x \in S$ s.t. $f(x) < g(x) + \varepsilon$

Let s be a simple function that minorises g

Define a simple function ε s.t. $f(x) < g(x) + \varepsilon(x)$

$f(x) - \varepsilon(x)$ is a simple function which minorises f and is strictly less than g

$\Rightarrow$ either S is a null set, or $\int_S f < \int_S g$

we have $\int_S f = \int_S g \Rightarrow S$ is a null set

$\therefore f = g$ almost everywhere