Homework 8

2) Let $f_n, f: \prod_{i=1}^{n}[a_i,b_i] \to \cancel{\mathbb{C}}\ \mathbb{R}$ s.t $f_n \to f$ a.e

Define $X(k,l) = \{x \in \prod_{i=1}^{n}[a_i,b_i] \mid \forall n \geq k, |f_n(x)-f(x)| < \frac{1}{l}\}$

$f_n \to f$ a.e ~~$\Rightarrow \forall x\ \exists k$ s.t~~

$\Rightarrow \forall \varepsilon > 0, \forall x, \exists k$ s.t $X(k, \frac{1}{\varepsilon}) \cup Z = [a,b]^n$, where Z is a zero set

$\Rightarrow \forall \varepsilon > 0 \quad \bigcup_k X(k, \frac{1}{\varepsilon}) \cup Z = [a,b]^n$, where Z is a zero set

RTP: $\forall \varepsilon > 0, \sigma > 0\ \exists X, k$ s.t $\forall x \in X, \forall n \geq k$, $|f_n(x)-f(x)| < \sigma$ and $m(X^c) < \varepsilon$

Let $\varepsilon, \sigma > 0$. Fix l s.t $\frac{1}{l} < \varepsilon$, $l \in \mathbb{N}$

By measure continuity, we have that,

$\lim_{k\to\infty} m(X(k,l)) = m(\lim_{k\to\infty} X(k,l)) = m(\prod[a_i,b_i])$

Choose $k_1 < k_2 < \ldots$ s.t ~~$m(X(k,l)^c) < \frac{\varepsilon}{2^l}$~~

$m(X(k_s,s)^c) < \frac{\varepsilon}{2^s} \quad \forall s \in \mathbb{N}$

Let $X = \bigcap_s X(k_s, s)$

$\Rightarrow m(X^c) \leq \sum_s \frac{\varepsilon}{2^s} < \varepsilon$

$\forall x \in X, \forall n \geq k_l$,

$x \in X \Rightarrow x \in X(k_l, l) \Rightarrow |f_n(x) - f(x)| < \frac{1}{l} < \sigma$

$\therefore X$ and k_l are s.t $m(X^c) < \varepsilon$ and $f_n \to f$ uniformly on X.

b) Egoroff's theorem holds. Let E be unbounded with finite measure. ~~We can~~ $\forall \varepsilon > 0$ we can find sufficiently large k s.t ~~E~~ $m(E \setminus [-k,k]^n) < \frac{\varepsilon}{2}$. Then our proof still stands for $[-k,k]^n$.

c) Let $f_n = \begin{cases} 1 & x \in [n, n+1] \\ 0 & \text{otherwise} \end{cases}$

$f_n \to 0$ almost everywhere but f_n does not converge uniformly ~~on~~

d) K is compact

$\Rightarrow$ K is closed and bounded

$\Rightarrow$ K has finite outer measure

~~$\Rightarrow K \cap S^c$ has finite~~

$\therefore$ We can apply Egoroff's theorem to f, f_n on K

3) $T: \mathbb{R}^n \to \mathbb{R}^n$ is defined by the matrix T_{ij}

$|(x_1, \ldots, x_n)| = \sum_{i \le n} x_i$

$\|T\| = \sup\{ \frac{|Tv|}{|v|} \mid |v| \neq 0 \} = \sup\{ |Tv| \mid |v| = 1 \}$

Let $T_{i,\cdot}$ denote the i^{th} collumn of T_{ij}

Let $|v| = 1$

$\Rightarrow |T(v)| = a_1|T_{1,\cdot}| + a_2|T_{2,\cdot}| + \ldots + a_n|T_{n,\cdot}|$ s.t $\sum a_i = 1$

$\Rightarrow \sup\{ |Tv| \mid |v| = 1 \} = \max(|T_{i,\cdot}|)$