Homework 9

1) Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$D_1 f(x,y) = \frac{y^3 - yx^2}{(x^2+y^2)^2} \quad D_2 f(x,y) = \frac{x^3 - xy^2}{(x^2+y^2)^2}$

$D_1 f(0,0) = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \to 0} \frac{\frac{0}{t^2} - 0}{t} = 0$

$D_2 f(0,0) = \lim_{t \to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t \to 0} \frac{\frac{0}{t^2} - 0}{t} = 0$

$\therefore D_1 f(x,y), D_2 f(x,y)$ exist $\forall (x,y) \in \mathbb{R}^2$

Let $S_n = \left(\frac{1}{n}, \frac{1}{n}\right)$

$S_n \to (0,0)$

However $f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{\frac{1}{n^2}}{\left(\frac{1}{n^2} + \frac{1}{n^2}\right)} = \frac{1}{2} \quad \forall n \in \mathbb{N}$

$\therefore \lim_{n \to \infty} f\left(\frac{1}{n}, \frac{1}{n}\right) \neq f\left(\lim_{n \to \infty}\left(\frac{1}{n}, \frac{1}{n}\right)\right)$

$\Rightarrow f$ is not continuous at $(0,0)$

2) $E \subset \mathbb{R}^n$ $f: E \to \mathbb{R}^m$ $D_1 f, \dots, D_n f$ bounded in E.

Let $M_i > D_i f \ \forall i \leq n$, $M = \sup_i M_i$

Let $\varepsilon > 0$. Define $\delta = \frac{\varepsilon}{Mn}$

Let $h = h_1 e_1 + \dots + h_n e_n$ be such that $|h| < \delta$

Let $V_0 = 0$, $V_1 = h_1 e_1, \dots, V_k = h_1 e_1 + \dots + h_k e_k, \dots, V_n = h$

$f(x+h) - f(x) = \sum_{i=1}^{n} f(x + V_i) - f(\cancel{x} + V_{i-1})$

By MVT:

$f(x+V_i) - f(x+V_{i-1}) = (D_i f)(x + V_{i-1} + \theta_i h_i e_i) h_i \quad \theta_i \in (0,1)$

$f(x+h) - f(x) = \sum_{i=1}^{n} (D_i f)(x + V_{i-1} + \theta_i h_i e_i) h_i \leq \sum_{i=1}^{n} M_i h_i$

$\leq M \sum_{i=1}^{n} h_i \leq M |h| \, n < Mn \frac{\varepsilon}{Mn} = \varepsilon$

$\therefore \forall \varepsilon > 0 \ \exists \delta$ s.t $|\cancel{f(x+h) - f(x)}| < \delta \Rightarrow |f(x+h) - f(x)| < \varepsilon$

$\Rightarrow f$ is continuous in E.

3) Let $E \subset \mathbb{R}^2$ be closed

Define $f:\mathbb{R}^2\to\mathbb{R}$ s.t $f(x) = \begin{cases} 0 & x\in E \\ \inf(|x-y| \mid y\in E) & x\notin E\end{cases}$

$\forall x_0 \in \mathbb{R}^2\setminus E \ \exists r>0$ s.t $\{x \mid |x_0-x|<r\} \subseteq \mathbb{R}^2\setminus E$

$\Rightarrow \inf(|x-y| \mid y\in E) > 0 \quad \forall x\in\mathbb{R}^2\setminus E$

$\therefore f^{-1}(E)=0$

Let $\varepsilon>0 \quad \delta<\varepsilon$

$\cancel{|x-y|<\delta \Rightarrow |g(x)-g(y)| = |\inf|x-z| - \inf|x}$

if $x,y\notin E \quad |x-y|<\delta \Rightarrow |g(x)-g(y)| < |x-y| < \delta<\varepsilon$

if $x,y\in E \quad |x-y|<\delta \Rightarrow |0-0|=0<\varepsilon$

if $x\notin E, y\in E \quad |x-y|<\delta \Rightarrow |g(x)-g(y)| = |g(x)|$

$= \inf(|x-y| \mid y\in E) < |x-y| < \delta<\varepsilon$

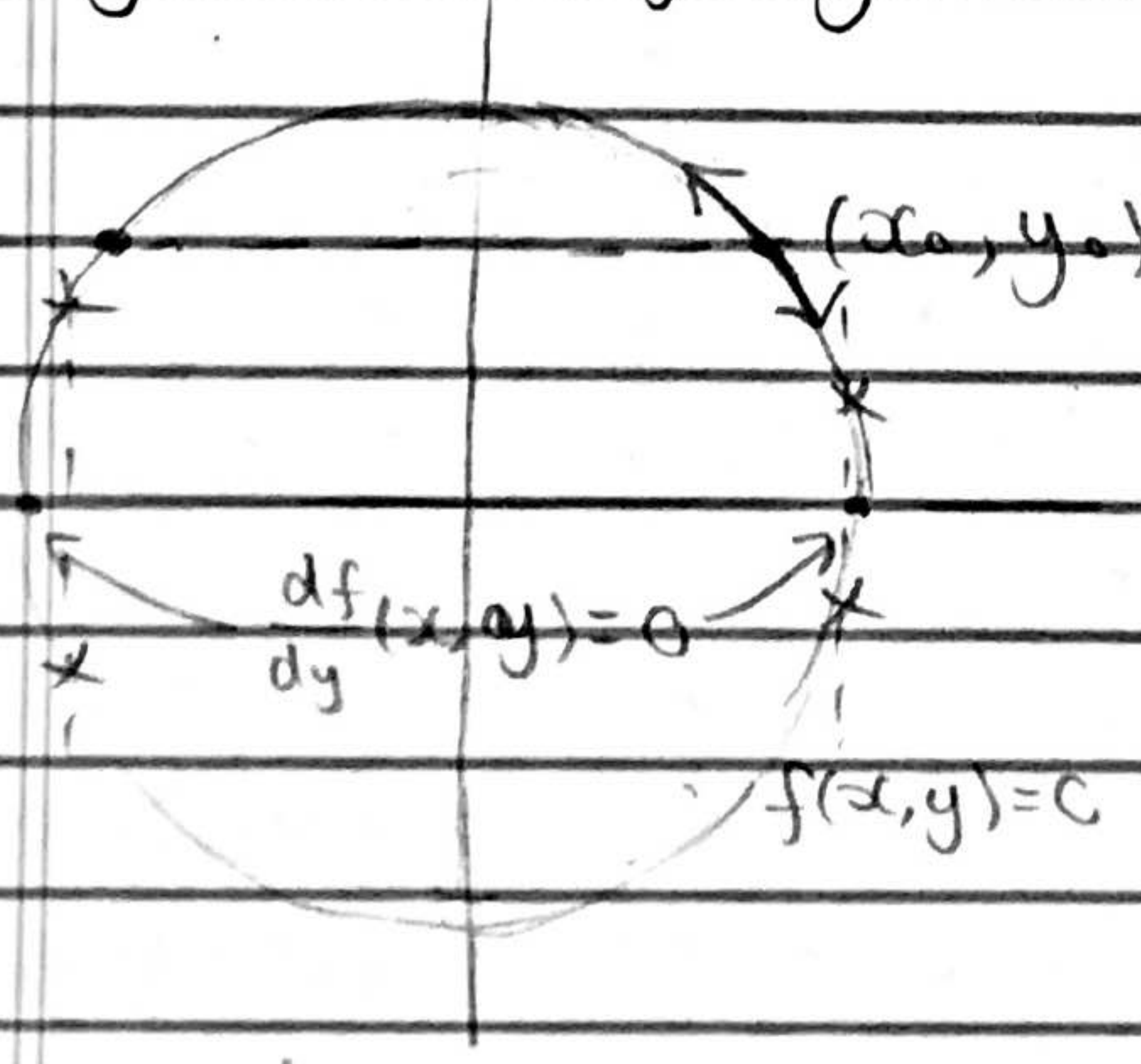
$\therefore f$ is continuous.

4) For $m=n=1$ the implicit function theorem states:

$\forall f:\mathbb{R}^2\to\mathbb{R}$, $(x_0,y_0)\in\mathbb{R}^2$ s.t $f(x_0,y_0)=\cancel{0}\ C\in\mathbb{R}$

if $\frac{df}{dy}(x_0,y_0)\neq 0$ then in the locality of (x_0,y_0), we can consider y as a continuous function of x.

eg. consider $f(x,y)=x^2+y^2$



In the region of (x_0,y_0) we can graph $f(x,y)=C$ as a function of x, avoiding the other possible y value with the local condition.

We can't do this at ~~(0,1~~ $(0,1)$ ~~(1,0~~ or $(0,-1)$ since $\frac{df}{dy}\cancel{\varepsilon}=0$.