Let $f: U \to \mathbb{R}^m$, $U \subset \mathbb{R}^n$

f is **differentiable** at $p \in U$ with derivative $(Df)_p = T$

if $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and

$$f(p+v) = f(p) + T(v) + R(v) \Rightarrow \lim_{|v| \to 0} \frac{R(v)}{|v|} = 0$$

$R(\cancel{v})$ is **sublinear** as it tends to 0 faster than $|v|$

If f is differentiable at p then its derivative can be unambiguously determined by

$$(Df)_p(u) = \lim_{t \to 0} \frac{f(p+tu) - f(p)}{t} \qquad \forall u \in \mathbb{R}^n$$

f is differentiable $\Rightarrow$ f is continuous

The **partial derivative** for ij of f at p is the limit

$$\frac{\partial f_i(p)}{\partial x_j} = \lim_{t \to 0} \frac{f_i(p+te_j) - f_i(p)}{t}$$

where $f_i(x)$ gives the i^{th} component of $f(x)$ and e_j is the orthonormal vector

If the partial derivatives of $f: U \to \mathbb{R}^m$ exist and are continuous then f is differentiable

Let f and g be differentiable

a) $D(f + cg) = Df + c\,Dg$

b) $D(c) = 0 \qquad D(T(x)) = T$

c) $D(g \circ f) = Dg \circ Df$ (chain rule)

d) $D(f \cdot g) = Df \cdot g + f \cdot Dg$ (Leibniz rule)

$f: U \to \mathbb{R}^m$ is differentiable at $p \in U$ iff each of its components f_i is differentiable at p

~~$(Df_i)_p =$~~ $D(f_i) = (Df)_i$

If $f: U \to \mathbb{R}^m$ is differentiable on U, $[p,q] \subset U$, then

$|f(q) - f(p)| \leq \sup\{\|(Df)_x\| \mid x \in U\}$

If $f: U \to \mathbb{R}^m$ has a continuous derivative, $[p,q] \subset U$,

$$f(q) - f(p) = T(q-p)$$

where T is the average derivative on the segment $[p,q]$,

$$T = \int_0^1 (Df)_{p+t(q-p)}\, dt$$

Let U be connected, $f: U \to \mathbb{R}^m$ differentiable

If $(Df)_x = 0 \ \forall x \in U$ then f is constant

Let $f: [a,b] \times (c,d) \to \mathbb{R}$ be continuous

If $\frac{\partial f(x,y)}{\partial (y)}$ exists and is continuous, then

$F(y) = \int_a^b f(x,y)\, dx$ has a continuous derivative, and

$$\frac{dF}{dy} = \int_a^b \frac{\partial f(x,y)}{\partial y}\, dx$$

If $f: \mathbb{R}^2 \to \mathbb{R}^m$ and $f(x_0, y_0) = z_0$, the z_0-locus of f is the set of points $(x,y) \in \mathbb{R}^2$ s.t $f(x,y) = z_0$

A C^r function has r continuous derivatives

Implicit function theorem
If the function f is C^r, $1 \leq r \leq \infty$, then near (x_0, y_0) where $f(x_0, y_0) = z_0$, the z_0-locus of f is the graph of a unique function $y = g(x)$, and g is a C^r function.

Inverse function theorem
If the derivative of f is invertible then f is a local diffeomorphism

A C^r diffeomorphism is a C^r bijection whose inverse is also C^r. Every diffeomorphism is a homeomorphism