Lebesgue measure

Riemann integrals give us a way of integrating certain functions $f: [a,b] \to \mathbb{R}$, where $[a,b]$ is an interval in $\mathbb{R}^n$

Lebesgue integrals allow us to integrate a wider class of functions $f: \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^n$ may not be an interval.

In order to compute lebesgue integrals we need a concept of the measure of Ω, $m(\Omega)$ which will act as an analogue for length, area and volume of intervals in $\mathbb{R}^n$, s.t $m(\Omega) = \int_\Omega 1$

Such a measure should satisfy certain properties such as the measure of a box corresponding to its area/volume, the measure of a set remaining constant under translation, $m(A) + m(B) = m(A \cup B)$ if A and B are disjoint, and $m(A) \leq m(B)$ if $A \subseteq B$.

Not all sets can be measured in this way, eg. the Banach Tarski paradox demonstrates that the set of points on the surface of a sphere can be decomposed and reassembled into 2 spheres by rotations, failing translational invariance.

A measurable set can be measured by a legesgue measure $m(\Omega)$ satisfying these properties.

Properties of the lesbegue measure:

Borel property: every open and every closed set in $\mathbb{R}^n$ is measurable

Complementarity: if Ω is measurable the $\mathbb{R}^n \setminus \Omega$ is also measurable

Boolean algebra: if $(\Omega_j)_{j \in J}$ is a finite collection of measurable then the union $\bigcup_{j \in J} \Omega_j$ and intersection $\bigcap_{j \in J} \Omega_j$ are also measurable

σ-algebra: if $(\Omega_j)_{j\in J}$ is a countable collection of measurable sets then the union $\bigcup_{j\in J}\Omega_j$ and the intersect $\bigcap_{j\in J}\Omega_j$ are also ~~countable~~ countable

Empty set: $m(\emptyset)=0$

Positivity: $0\leq m(\Omega)\leq +\infty$ for every measurable set Ω

Monotonicity: If A and B are measurable and $A\subseteq B$ then $m(A)\leq m(B)$

Finite subadditivity: If $(A_j)_{j\in J}$ ~~are~~ is a finite collection of measurable sets then $m(\bigcup_{j\in J}A_j)\leq \sum_{j\in J} m(A_j)$

Finite additivity: If $(A_j)_{j\in J}$ is a finite collection of disjoint measurable sets then $m(\bigcup_{j\in J}A_j)=\sum_{j\in J} m(A_j)$

Countable subadditivity: If $(A_j)_{j\in J}$ is a countable collection of measurable sets then $m(\bigcup_{j\in J}A_j)\leq \sum_{j\in J} m(A_j)$

Countable additivity: If $(A_j)_{j\in J}$ is a countable collection of disjoint measurable sets then $m(\bigcup_{j\in J}A_j)=\sum_{j\in J} m(A_j)$

Normalisation: The unit cube $[0,1]^n=\{(x_1,\dots x_n)\in\mathbb{R}^n \mid 0\leq x_i\leq 1\ \forall i\leq n\}$ has measure $m([0,1]^n)=1$

Translational invariance: If Ω is a measurable set and $x\in\mathbb{R}^n$ then $x+\Omega=\{x+y \mid y\in\Omega\}$ is measurable and $m(x+\Omega)=m(\Omega)$

There exists a measure which satisfies the above properties and can assign a number $m(\Omega)$ to every measurable set Ω

Outer Measure

An open box $B \subseteq \mathbb{R}^n$ is a set of the form:

$$B = \prod_{i=1}^{n} (a_i, b_i) = \{(x_1, \dots, x_n) \mid x_i \in (a_i, b_i) \ \forall i \leq n\}$$

This is an open interval in $\mathbb{R}$, an open ~~sq~~ rectangle in $\mathbb{R}^2$ etc.

A collection of boxes $(B_j)_{j\in J}$ cover $\Omega \subseteq \mathbb{R}^n$ if $\Omega \subseteq \bigcup_{j\in J} B_j$

The outer measure of a set Ω is the quantity

$$m^*(\Omega) = \inf\{\sum_{j\in J} vol(B_j) \mid (B_j)_{j\in J} \text{ is a countable collection of open boxes which cover } \Omega\}$$

Properties of the outer measure:
- → $m^*(\emptyset) = 0$
- → $0 \leq m^*(\Omega) \leq +\infty \quad \forall \Omega$
- → monotonicity
- → finite subadditivity
- → countable subadditivity

For any closed box $B = \prod_{i=1}^{n} [a_i, b_i] = \{(x_1, \dots, x_n) \mid x_i \in [a_i, b_i] \ \forall i \leq n\}$

$$m^*(B) = \prod_{i=1}^{n} (b_i - a_i)$$

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$$m^*(B) = \prod_{i=1}^{n} (b_i - a_i)$$

$m^*(\mathbb{R}) = +\infty \quad m^*(\mathbb{Q}) = 0 \quad m^*(\mathbb{R}\setminus\mathbb{Q}) = +\infty$

$m^*(\{x \in \mathbb{R} \mid x \in (0,1)\}) = 1 \quad m^*(\{(x,0) \in \mathbb{R}^2 \mid x \in (0,1)\}) = 0$

There exists a countable collection of disjoint sets $(A_j)_{j\in J} \subseteq \mathbb{R}$ s.t $m^*(\bigcup_{j\in J} A_j) \neq \sum_{j\in J} \cancel{A_j}\, m^*(A_j)$

There exists a finite collection of disjoint sets $(A_j)_{j\in J} \subseteq \mathbb{R}$ s.t $m^*(\bigcup_{j\in J} A_j) \neq \sum_{j\in J} m^*(A_j)$

Therefore the outer measure fails both finite and countable additivity

Measurable sets

A set $E \subseteq \mathbb{R}^n$ is **Lebesgue measurable** if $\forall A \subseteq \mathbb{R}^n$,
$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$$

~~At~~ The **half space** is the set $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \cancel{x_n}\ x_n > 0\}$
Half spaces are measurable

Properties of ~~half spaces~~ measurable sets:
a) E is measurable $\Rightarrow \mathbb{R}^n \setminus E$ is measurable
b) Translation invariance
c) ~~If~~ E_1 and E_2 are measurable $\Rightarrow E_1 \cap E_2$ and $E_1 \cup E_2$ are measurable
d) Boolean algebra property
e) Every open and every closed box is measurable (Borel)
f) Any set of outer measure 0 is measurable
g) Finite additivity
h) If $A \subseteq B$ are measurable the $B \setminus A$ is measurable and $m(B \setminus A) = m(B) - m(A)$
i) Countable additivity
~~If~~

Every open set can be written as a countable or finite union of open boxes

Let $\{E_k\}$ and $\{F_k\}$ be sequences of measurable sets
$E_k \uparrow E$ indicates $E_1 \subset E_2 \subset \dots$ and $E = \cup E_k$
$F_k \downarrow F$ indicates $F_1 \supset F_2 \supset \dots$ and $F = \cap F_k$

Upward measure continuity: $E_k \uparrow E \Rightarrow m(E_k) \uparrow m(E)$
Downward measure continuity: $F_k \downarrow F,\ m(F_1) < \infty \Rightarrow m(F_k) \downarrow m(F)$

Measurable functions

Let Ω be a measurable set in $\mathbb{R}^n$ and let $f: \Omega \to \mathbb{R}^m$ be a function. f is measurable iff $f^{-1}(V)$ is measurable for every open set $V \subseteq \mathbb{R}^m$

If $\Omega \subseteq \mathbb{R}^n$ is measurable and $f: \Omega \to \mathbb{R}^m$ is continuous then f is measurable

Let $\Omega \subseteq \mathbb{R}^n$ be measurable and let $f: \Omega \to \mathbb{R}^m$
f is measurable iff $f^{-1}(B)$ is measurable $\forall$ open box B

Let $f: \Omega \to \mathbb{R}^n$ s.t $f = (f_1, \dots, f_i, \dots, f_n)$, $f_i: \Omega \to \mathbb{R}$
f is measurable iff f_i is measurable $\forall i$

Let $\Omega \subseteq \mathbb{R}^n$ be measurable and let $W \subseteq \mathbb{R}^m$ be an open set
If $f: \Omega \to W$ is measurable and $g: W \to \mathbb{R}^p$ is continuous then $g \circ f: \Omega \to \mathbb{R}^p$ is measurable

If $f: \Omega \to \mathbb{R}$ and $g: \Omega \to \mathbb{R}$ are measurable functions, then so are $f+g$, $f-g$, fg, $\max(f,g)$, $\min(f,g)$
If $g(x) \neq 0 \ \forall x \in \Omega$, then so is $\frac{f}{g}$

Let $\Omega \subseteq \mathbb{R}^n$ be measurable and $f: \Omega \to \mathbb{R}^m$
f is measurable iff $f^{-1}(a, \infty)$ is measurable $\forall a \in \mathbb{R}$

Let Ω be a measurable subset of $\mathbb{R}^n$, $f: \Omega \to \mathbb{R}^*$
f is measurable iff $f^{-1}(a, +\infty]$ is measurable $\forall a \in \mathbb{R}$

Let Ω be measurable, let $f_n: \Omega \to \mathbb{R}^m$ be measurable $\forall n$
The functions $\sup f_n$, $\inf f_n$, $\limsup f_n$ and $\liminf f_n$ are also measurable
If $f_n \to f$ ~~$\Omega \to \mathbb{R}$~~ pointwise, then f is measurable

Abstract Measure Spaces

Let S be a set, 2^S be the power set of S

A **σ-algebra** is a subset $\mathcal{M}_S \subset 2^S$ satisfying:

1) $\emptyset \in \mathcal{M}_S$
2) $\mathcal{M}_S$ is closed under countable unions ($A_i \in \mathcal{M}_S \Rightarrow \bigcup A_i \in \mathcal{M}_S$)
3) $\mathcal{M}_S$ is closed under complementation ($A \in \mathcal{M}_S \Rightarrow A^c \in \mathcal{M}_S$)

A space with a σ-algebra $(S, \mathcal{M}_S)$ is a **measurable space**

A **measure** on $(S, \mathcal{M}_S)$ is a function $\omega : \mathcal{M}_S \to [0, \infty]$ s.t:

1) $\omega(\emptyset) = 0$
2) countable additivity holds for disjoint sets in $\mathcal{M}_S$
   $\omega(\bigcup A_i) = \sum \omega(A_i)$

A space with a σ-algebra and a measure $(S, \mathcal{M}_S, \omega)$ is a **measure space**

A **topological space** $(S, \mathcal{T}_S)$ is a set S and a collection of subsets $\mathcal{T}_S \subset 2^S$, satisfying:

1) $\emptyset, S \in \mathcal{T}_S$
2) $\mathcal{T}_S$ is closed under finite intersection
3) $\mathcal{T}_S$ is closed under arbitrary unions

The elements of $\mathcal{T}_S$ are called **open sets**. A set $X \subset S$ is **closed** iff $X^c \in \mathcal{T}_S$. $\mathcal{T}_S$ is a **topology** on S.

If $(S, \mathcal{T}_S)$ is a topological space, there exists a minimal σ-algebra containing $\mathcal{T}_S$ called a **Borel σ-algebra** denoted $\mathcal{B}_S$. A **Borel set** is a subset of S in $\mathcal{B}_S$.

If $(S, \mathcal{T}_S)$ is a topological space, then a countable intersection of open sets is a **G_δ-set** and a countable union of closed sets is an **F_σ-set**.
G_δ-sets and F_σ-sets are Borel sets.

Let S be a set. Let ω be an outer measure on S
An outer measure is a function $\omega : 2^S \to [0,\infty]$ satisfying:
1) $\omega(\emptyset) = 0$
2) ~~AC~~ Monotonicity $(A \subset B \Rightarrow \omega(A) \leq \omega(B))$
3) Countable subadditivity $(\omega(\bigcup A_i) \leq \sum \omega(A_i))$
We can define $M_S \subset 2^S$ s.t ~~E~~ a set $E \subset S$ is in M_S if
$\forall X \subset S, \ \omega(X) = \omega(X \cap E) + \omega(X \cap E^c)$
M_S is a σ-algebra and (S, M_S, ω) is a measure space.

A null set is a subset $E \subset S$ with $\omega(E) = 0$

If $E \subset S$ is a null set:
1) $\forall A \subset S \quad \omega(A \cup E) = \omega(A)$
2) $E' \subset E \Rightarrow \omega(E') = 0$
3) $\forall A \subset S \quad \omega(A \cap E^c) = \omega(A)$
4) $\omega(E) = 0 \Rightarrow E$ is measurable
5)

The outer measure ω is unchanged by adding or removing a null set E
Every lebegue measurable set in S is equivalent to a Borel set plus or minus a null set

$\forall E \subset S$, E is lebesgue measurable iff $\exists$ an F_σ-set F and a G_δ-set G s.t $F \subset E \subset G$ and $m(G \setminus F) = 0$

Open sets and closed sets in $\mathbb{R}^n$ are Lebesgue measurable

The Lebesgue measure of a closed or partially closed box is the volume of its interior
The boundary of a box is a null set.

Lebesgue measure is regular: $\forall$ measurable set E, $\exists$ F_σ-set F and G_δ-set G s.t $F \subset E \subset G$ and $G \setminus F$ is a null set.

A bounded subset $E \subset \mathbb{R}^n$ is measurable iff $\exists$ F_σ-set F and G_δ-set G s.t $F \subset E \subset G$ and $m(F) = m(G)$

Every open set is a countable disjoint union of balls, plus a null set
Every open set in $\mathbb{R}^n$ is a countable disjoint union of open boxes plus a null set

Let E be a measurable set. $\exists$ G_δ-set H_E and F_σ-set K_E s.t $K_E \subset E \subset H_E$ and $m(H_E \setminus K_E) = 0$
H_E is the hull of E. K_E is the kernel of E.
These are unique up to null sets.
The outer measure $m^*(E)$ is attained by the hull H_E
The inner measure $m_*(E)$ is attained by the kernel K_E

If $E \subset \mathbb{R}^n$, $F \subset \mathbb{R}^k$ are measurable sets then
$$m(E \times F) = m(E)\,m(F)$$
By convention, if $m(E) = \infty$ and $m(F) = 0$, $m(E \times F) = 0$

The hull of a product is the product of the hulls
The kernel of a product is the ~~kernel~~ product of the kernels

A slice of a set $E \subset \mathbb{R}^n \times \mathbb{R}^k$ at $x \in \mathbb{R}^n$ is the set,
$$E_x = \{ y \in \mathbb{R}^k \mid (x, y) \in E \} \subseteq \mathbb{R}^k$$

Zero slice theorem:
If $E \subset \mathbb{R}^n \times \mathbb{R}^k$ is measurable then E is a zero set iff almost every slice of E is a zero set.

Let $W \subset I^n$ be a set. $X_\alpha(W) = \{ x \mid m(W_x) > \alpha \}$
We can think of the slices W_x, $x \in X_\alpha$ as 'heavy'.

If $W \subset I^n$ is open then $m(W) \geq m(X_\alpha(W))\,\alpha$