7.2.1.

(v) $\emptyset$ is covered by the empty collection, whose sum of volumes is 0.

(vi)

Every open box has non-negative real volume,
so every sum

$$\sum_{i \in I} \operatorname{vol}(A_i)$$

(for open boxes A_i) is a non-negative extended real number,
so for any $\Omega \subseteq \mathbb{R}^n$

$$\left\{ \sum_{i \in I} \operatorname{vol}(A_i) \,\middle|\, I \text{ countable}, \bigcup_{i \in I} A_i \supseteq \Omega \right\}$$

contains only non-negative extended real values, hence its infimum $m^*(\Omega)$ is a non-negative extended real number.

(vii)

For any $\varepsilon > 0$
there's a countable cover $\mathcal{U}$ of B by open boxes
such that

$$\sum_{U \in \mathcal{U}} \operatorname{vol}(U) < m^*(B) + \varepsilon$$

Since $A \subset B, \mathcal{U}$ covers A, giving

$$m^*(A) \leq \sum_{U \in \mathcal{U}} \operatorname{vol}(U) < m^*(B) + \varepsilon$$

for all $\varepsilon > 0$, or simply

$$m^*(A) \leq m^*(B)$$

(viii) (x).

(x)

Let $\varepsilon > 0$, $\sum_{j \in J} \varepsilon_j = \varepsilon$, and $A = \bigcup_{j \in J} A_j$.
For each $j \in J$, pick a countable cover $\mathcal{U}_|$ of A_j by open boxes such that

$$\sum_{U \in \mathcal{U}_j} \operatorname{vol}(U) < m^*(A_j) + \varepsilon_j$$

Then

$$\mathcal{U} \coloneqq \bigcup_{j \in J} \mathcal{U}_j$$

(which is countable) covers A, hence

$$\begin{aligned} m^*(A) &\leq \sum_{U \in \mathcal{U}} \operatorname{vol}(U) \\ &\leq \sum_{j \in J} \sum_{U \in \mathcal{U}_J} \\ &< \sum_{j \in J} m^*(A_j) + \varepsilon_j \\ &= \left(\sum_{j \in J} m^*(A_j) \right) + \varepsilon \end{aligned}$$

for any $\varepsilon > 0$, giving

$$m^*(A) \leq \sum_{j \in J} m^*(A_j)$$

(xiii)

Volume (of an open box) is translation invariant,
so any cover $\mathcal{U}$ of $A \subseteq \mathbb{R}^n$ by open boxes
has the same sum of volumes as $\mathcal{U} + x$ (which covers $A + x$),
hence

$$\begin{aligned} &\left\{ \sum_{U \in \mathcal{U}} \mathrm{vol}(U) \;\middle|\; \mathcal{U} \text{ a countable cover of } A \text{ by open boxes} \right\} \\ = &\left\{ \sum_{U \in \mathcal{U}} \mathrm{vol}(U) \;\middle|\; \mathcal{U} \text{ a countable cover of } A + x \text{ by open boxes} \right\} \end{aligned}$$

so $m^*(A) = m^*(A + x)$.

7.2.2.

Let

$$A \subseteq \mathbb{R}^m$$
$$B \subseteq \mathbb{R}^n$$

(This is different from Tao's choice because I like alphabetical order.)

Definition. *Let $S \subseteq \mathbb{R}^k$ for some k and let $\mathcal{U}_S$ be a collection of subsets of $\mathbb{R}^k$. $\mathcal{U}_S$ is **suitable** if it is a countable cover of S by open boxes.*

Given two open boxes $U_A \subseteq \mathbb{R}^m, U_B \subseteq \mathbb{R}^n$,

$$\mathrm{vol}_{m+n}(U_A \times U_B) = \mathrm{vol}_m(U_A) \cdot \mathrm{vol}_n(U_B)$$

Given two suitable covers $\mathcal{U}_A, \mathcal{U}_B$, we can form the collection

$$\mathcal{U}_{A\times B} = \{U_A \times U_B \mid U_A \in \mathcal{U}_A, U_B \in \mathcal{U}_B\}$$

Lemma 2: $\mathcal{U}_{A\times B}$ *is suitable.*

Clearly the elements of $\mathcal{U}_{A\times B}$ are open boxes in $\mathbb{R}^{m+n}$.

Furthermore, $\mathcal{U}_{A\times B}$ is countable because there exist injective functions:

$$\mathcal{U}_{A\times B} \hookrightarrow \mathcal{U}_A \times \mathcal{U}_B \hookrightarrow \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$$

In particular,

$$\mathcal{U}_{A\times B} \hookrightarrow \mathcal{U}_A \times \mathcal{U}_B : U_A \times U_B \mapsto (U_A, U_B)$$
$$\mathcal{U}_A \times \mathcal{U}_B = \iota_A \times \iota_B$$

where $\iota_A : \mathcal{U}_A \hookrightarrow \mathbb{N}$ and similar for B.

Finally, $\mathcal{U}_{A\times B}$ covers $A \times B$ because for any $(a, b) \in A \times B$, there exist $U_A \in \mathcal{U}_A, U_B \in \mathcal{U}_B$ such that $a \in U_A, b \in U_B$, hence

$$(a, b) \in U_A \times U_B \in \mathcal{U}_{A\times B}$$

Now note that

$$\sum_{U\in\mathcal{U}_{A\times B}} \mathrm{vol}_{m+n}(U) = \sum_{(U_A,U_B)\in\mathcal{U}_A\times\mathcal{U}_B} \mathrm{vol}_m(U_A)\,\mathrm{vol}_n(U_B)$$
$$= \left(\sum_{U_A\in\mathcal{U}_A} \mathrm{vol}_m(U_A)\right)\left(\sum_{U_B\in\mathcal{U}_B} \mathrm{vol}_m(U_B)\right)$$

Now, letting

$$\sum_{U_A\in\mathcal{U}_A} \mathrm{vol}_m(U_A) < m_m^*(A) + \varepsilon_A$$
$$\sum_{U_B\in\mathcal{U}_B} \mathrm{vol}_n(U_B) < m_n^*(B) + \varepsilon_B$$

for some $\varepsilon_A, \varepsilon_B > 0$, we have

$$m_{m+n}^*(A \times B) \le \sum_{U\in\mathcal{U}_{A\times B}} \mathrm{vol}_{m+n}(U)$$
$$= \left(\sum_{U_A\in\mathcal{U}_A} \mathrm{vol}_m(U_A)\right)\left(\sum_{U_B\in\mathcal{U}_B} \mathrm{vol}_m(U_B)\right)$$
$$< (m_m^*(A) + \varepsilon_A)\,(m_n^*(B) + \varepsilon_B)$$
$$= m_m^*(A)m_n^*(B) + \varepsilon_A m_n^*(B) + \varepsilon_B m_m^*(A) + \varepsilon_A\varepsilon_B$$

(The first inequality comes from the suitability of $\mathcal{U}_{A\times B}$.)
Since $\varepsilon_A, \varepsilon_B$ can be arbitrarily small, we find

$$m^*_{m+n}(A \times B) \leq m^*_m(A)m^*_n(B)$$

7.2.3.

(a)

We disjointize the sequence. Let

$$D_i = A_i \setminus \bigcup_{j<i} A_j$$

Then D_i are disjoint and

$$A_i = \bigcup_{j \le i} D_j$$

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} D_i$$

By finite additivity,

$$m(A_i) = \sum_{j=1}^{i} m(D_j)$$

By countable additivity,

$$\begin{aligned} m(\bigcup_{j=1}^{\infty} A_j) &= \sum_{j=1}^{\infty} m(D_j) \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} m(D_j) \\ &= \lim_{n\to\infty} m(A_n) \end{aligned}$$

(b)

Again, we disjointize:

$$D_i = A_i \setminus A_{i+1}$$

For any i,

$$A_i = \bigcap_{j=1}^{\infty} A_j \cup \bigcup_{j=i}^{\infty} D_j$$

$$\infty > m(A_1) \ge m(A_i) = m(\bigcap_{j=1}^{\infty} A_j) + \sum_{j=i}^{\infty} m(D_j)$$

Letting $i = 1$, we notice that

$$\sum_{j=1}^{\infty} m(D_j)$$

converges in $\mathbb{R}$, hence

$$\lim_{i\to\infty} \sum_{j=i}^{\infty} m(D_j) = 0$$

Now, from the equation

$$m(A_i) = m(\bigcap_{j=1}^{\infty} A_j) + \sum_{j=i}^{\infty} m(D_j)$$

we obtain

$$\lim_{i\to\infty} m(A_i) = m(\bigcap_{j=1}^{\infty} A_j) + \lim_{i\to\infty} \sum_{j=i}^{\infty} m(D_j)$$

$$\lim_{i\to\infty} m(A_i) = m(\bigcap_{j=1}^{\infty} A_j) + 0$$

$$\lim_{i\to\infty} m(A_i) = m(\bigcap_{j=1}^{\infty} A_j)$$

QED.

7.2.4.

Use open cubes $(0, \frac{1}{q})^n$ to tile the unit cube, creating the set

$$O = \left\{ \prod_{i=1}^{n} (\frac{p_i - 1}{q}, \frac{p_i}{q}) \middle| p_i \in \{1, \dots q\} \right\}$$

These q^n cubes have equal measure by translation invariance.
They are disjoint.
Furthermore,

$$\bigcup O \subseteq [0,1]^n$$

so

$$\begin{aligned} q^n m((0, \frac{1}{q})^n) &= \sum_{o \in O} m(o) \\ &= m(\bigcup O) \\ &\leq m([0,1]^n) \\ &= 1 \end{aligned}$$

$$m((0, \frac{1}{q})^n) \leq \frac{1}{q^n}$$

Similarly, let

$$C = \left\{ \prod_{i=1}^{n} [\frac{p_i - 1}{q}, \frac{p_i}{q}] \middle| p_i \in \{1, \dots q\} \right\}$$

These q^n closed cubes have equal measure by translation invariance.
They cover the unit cube:

$$[0,1]^n \subseteq \bigcup C$$

This gives

$$\begin{aligned} 1 &= m([0,1]^n) \\ &\leq \sum_{c \in C} m(c) \\ &= q^n m([0, \frac{1}{q}]^n) \end{aligned}$$

$$\frac{1}{q^n} \leq m([0, \frac{1}{q}]^n)$$

Now we just need to show that $(0, \frac{1}{q})^n$ and $[0, \frac{1}{q}]^n$ have the same measure, giving

$$\frac{1}{q^n} \leq m([0, \frac{1}{q}]^n) = m((0, \frac{1}{q})^n) \leq \frac{1}{q^n}$$

$$\frac{1}{q^n} = m([0, \frac{1}{q}]^n) = m((0, \frac{1}{q})^n)$$

Claim: $m([0, \frac{1}{q}]^n) = m((0, \frac{1}{q})^n)$

For each $i \in \{1, \dots n\}$ and $x \in \mathbb{R}$, define

$$F_{ix} = [0, \frac{1}{q}] \times \cdots \times \{x\} \times \cdots [0, \frac{1}{q}]$$

where the Cartesian product has n factors and $\{x\}$ is the i^{th} factor. Clearly

$$[0, \frac{1}{q}]^n \setminus (0, \frac{1}{q})^n = \bigcup_{i=1}^{n} (F_{i0} \cup F_{i1})$$

E.g. in $\mathbb{R}^3$, the above set is the union of the six faces of a cube.

Fix $i \in \{1, \dots n\}$ and $j \in \{0, 1\}$. By translation invariance,

$$\sum_{x \in \mathbb{Q} \cap [0,1]} m(F_{ij}) = \sum_{x \in \mathbb{Q} \cap [0,1]} m(F_{ix})$$

Since $\mathbb{Q} \cap [0, 1]$ is infinite,

$$\sum_{x \in \mathbb{Q} \cap [0,1]} m(F_{ij}) \in \{0, \infty\}$$

Noting that these F_{ix} are disjoint, countable additivity gives

$$\sum_{x \in \mathbb{Q} \cap [0,1]} m(F_{ix}) = m(\bigcup_{x \in \mathbb{Q} \cap [0,1]} F_{ix})$$

Noting that

$$\bigcup_{x \in \mathbb{Q} \cap [0,1]} F_{ix} \subseteq [0, 1]^n$$

monotonicity gives

$$\sum_{x \in \mathbb{Q} \cap [0,1]} m(F_{ix}) \leq 1$$

Recalling that the sum is either 0 or ∞, we have

$$m(\bigcup_{x \in \mathbb{Q} \cap [0,1]} F_{ix}) = 0$$

$$m(F_{ij}) = 0$$

$$\begin{aligned} m([0, \frac{1}{q}]^n \setminus (0, \frac{1}{q})^n) &= m(\bigcup_{i=1}^{n} (F_{i0} \cup F_{i1})) \\ &\leq \sum_{i=1}^{n} (0 + 0) \\ &= 0 \end{aligned}$$

So, by finite additivity we have

$$\begin{aligned} m([0, \frac{1}{q}]^n) &= m((0, \frac{1}{q})^n) + m([0, \frac{1}{q}]^n \setminus (0, \frac{1}{q})^n) \\ &= m((0, \frac{1}{q})^n) + 0 \\ &= m((0, \frac{1}{q})^n) \end{aligned}$$

This proves the claim.

QED.

7.4.1. Special case of 7.4.2.

7.4.2.

Let

$$A = \prod_{i=1}^{n} (a_i, b_i)$$

and note that

$$E = \mathbb{R}^{n-1} \times (0, \infty)$$

hence

$$A \cap E = \prod_{i=1}^{n-1} (a_i, b_i) \times ((a_n, b_n) \cap (0, \infty))$$

$$A \setminus E = \prod_{i=1}^{n-1} (a_i, b_i) \times ((a_n, b_n) \setminus (0, \infty))$$

Trivial case.

If one of

$$(a_n, b_n) \cap (0, \infty)$$

$$(a_n, b_n) \setminus (0, \infty)$$

is empty, then the other one is (a_n, b_n), giving

$$m^*(A \cap E) = m^*(A \setminus E) = m^*(\emptyset) + m^*(A) = m^*(A)$$

as was to be shown.

If the above case fails, then

$$a_n < 0 < b_n$$

This gives

$$A \cap E = \prod_{i=1}^{n-1} (a_i, b_i) \times (0, b_n)$$

$$A \setminus E = \prod_{i=1}^{n-1} (a_i, b_i) \times (a_n, 0]$$

which allows us to determine that

$$m^*(A \cap E) = \prod_{i=1}^{n-1} (b_i - a_i) \cdot b_n$$

We also determine (using monotonicity) that

$$\begin{aligned} \prod_{i=1}^{n-1} (b_i - a_i) \cdot (-a_n) &= m^*(\prod_{i=1}^{n-1} (a_i, b_i) \times (a_n, 0)) \\ &\leq m^*(A \setminus E) \\ &= m^*(\prod_{i=1}^{n-1} (a_i, b_i) \times [a_n, 0]) \\ &= \prod_{i=1}^{n-1} (b_i - a_i) \cdot (-a_n) \end{aligned}$$

$$m^*(A \setminus E) = \prod_{i=1}^{n-1} (b_i - a_i) \cdot (-a_n)$$

Which finally yields

$$\begin{aligned} m^*(A \cap E) + m^*(A \setminus E) &= \prod_{i=1}^{n-1} (b_i - a_i) \cdot b_n + \prod_{i=1}^{n-1} (b_i - a_i) \cdot (-a_n) \\ &= \prod_{i=1}^{n-1} (b_i - a_i) \cdot (b_n - a_n) \\ &= \prod_{i=1}^{n} (b_i - a_i) \\ &= m^*(A) \end{aligned}$$

QED.

7.4.3.

Let $A \subseteq \mathbb{R}^n$.
By finite sub-additivity,

$$m^*(A \cap E) + m^*(A \setminus E) \geq m^*(A)$$

We now prove $\leq$.
Pick an $\varepsilon > 0$ and let $(U_k)_{k \in K}$ be a countable open box cover of A with

$$\sum_{k \in K} m^*(U_k) < m^*(A) + \varepsilon$$

Note that

$$A \cap E \subseteq \bigcup_{k \in K} (U_k \cap E) \qquad m^*(A \cap E) \leq \sum_{k \in K} m^*(U_k \cap E)$$
$$A \setminus E \subseteq \bigcup_{k \in K} (U_k \setminus E) \qquad m^*(A \setminus E) \leq \sum_{k \in K} m^*(U_k \setminus E)$$

Now we find

$$\begin{aligned} m^*(A \cap E) + m^*(A \setminus E) &\leq \sum_{k \in K} m^*(U_k \cap E) + \sum_{k \in K} m^*(U_k \setminus E) \\ &= \sum_{k \in K} \left(m^*(U_k \cap E) + m^*(U_k \setminus E) \right) \\ &\overset{7.4.2}{=} \sum_{k \in K} m^*(U_k) \\ &< m^*(A) + \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary,

$$m^*(A \cap E) + m^*(A \setminus E) \leq m^*(A)$$

Combining with our initial inequality, we obtain

$$m^*(A \cap E) + m^*(A \setminus E) = m^*(A)$$

Since $A \subseteq \mathbb{R}^n$ was arbitrary, E is measurable.

7.4.4.

(a)

Denoting $\mathbb{R}^n \setminus E$ by E^c, we have

$$\begin{aligned} m^*(A) &= m^*(A \cap E) + m^*(A \setminus E) \\ &= m^*(A \setminus E^c) + m^*(A \cap E^c) \end{aligned}$$

(b)

We use the translation invariance of m^*.

$$\begin{aligned} m^*(A) &= m^*(-x + A) \\ &= m^*((-x + A) \cap E) + m^*((-x + A) \setminus E) \\ &= m^*(x + ((-x + A) \cap E)) + m^*(x + ((-x + A) \setminus E)) \\ &= m^*(A \cap (x + E)) + m^*(A \setminus (x + E))) \end{aligned}$$

(The last $=$ comes from equality of sets due to the translation invariance of $\cap$ and $\setminus$.)

(c)

Unions:

First, for free, we get

$$m^*(A) \leq m^*(A \cap (E_1 \cup E_2)) + m^*(A \setminus (E_1 \cup E_2))$$

by finite sub-additivity of m^*. Now we prove $\geq$.

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*(A \setminus E_1) \\ &= m^*(A \cap E_1) + m^*((A \setminus E_1) \cap E_2) + m^*((A \setminus E_1) \setminus E_2) \end{aligned}$$

By finite sub-additivity,

$$m^*(A \cap E_1) + m^*((A \setminus E_1) \cap E_2) \geq m^*(A \cap (E_1 \cup E_2))$$

Noting that

$$(A \setminus E_1) \setminus E_2 = A \setminus (E_1 \cup E_2)$$

We now have

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*((A \setminus E_1) \cap E_2) + m^*((A \setminus E_1) \setminus E_2) \\ &\geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \setminus (E_1 \cup E_2)) \end{aligned}$$

This proves that $E_1 \cup E_2$ is measurable.

Intersections:

$$E_1 \cap E_2 = (E_1^c \cup E_2^c)^c$$

(d) Induction.

(e)

Given $i \in \{1, \ldots n\}$ and $x \in \mathbb{R}$, we write P_{ix} to denote the half-plane above x in the i^{th} dimension,

$$P_{ix} = \{(p_1, \ldots p_n) | p_i > x\}$$

and P^{ix} to denote the half-plane below x in the i^{th} dimension,

$$P^{ix} = \{(p_1, \dots p_n) | p_i < x\}$$

Now, taking an open box

$$B = \prod_{i=1}^{n} (a_i, b_i)$$

we find

$$B = \bigcap_{i=1}^{n} \left(P_{ia_i} \cap P^{ib_i}\right)$$

which is measurable by (c) and 7.4.3.

(f)

By finite sub-additivity,

$$m^*(A \cap E) + m^*(A \setminus E) \geq m^*(A)$$

Furthermore,

$$\begin{aligned} m^*(A \cap E) + m^*(A \setminus E) &\leq m^*(E) + m^*(A \setminus E) \\ &= 0 + m^*(A \setminus E) \\ &= m^*(A \setminus E) \\ &\leq m^*(A) \end{aligned}$$

So

$$m^*(A \cap E) + m^*(A \setminus E) = m^*(A)$$