Lemma 0.

Let
$$d = \text{dist}(E, F)$$
One easily sees that a closed box with side lengths less than
$$s := \frac{d}{\sqrt{n}}$$
has diameter $< d$.
Note that, for any $A \subseteq \mathbb{R}^n$,
$$m^*(A) = \inf \left\{ \sum_{j\in J} |B_j| \,\middle|\, J \text{ countable, } B_j \text{ closed boxes} \right\}$$
Now let $(B_j)_{j\in J}$ be a countable closed box cover of $E \cup F$ such that
$$\sum_{j\in J} |B_j| < m^*(E \cup F) + \varepsilon$$
for some arbitrary $\varepsilon > 0$ Write
$$B_j = \prod_{i=1}^{n} [a_{ij}, b_{ij}]$$
for each $j \in J$, and partition each interval as
$$P_{ij} = \left\{a_{ij} = x_{ij0} < \cdots < x_{ijm_{ij}} = b_{ij}\right\}$$
with $x_{ijk} - x_{ij(k-1)} < s$. This allows us to cover B_j with sub-boxes of the form:
$$b_{jk_1k_2\ldots k_n} = \prod_{i=1}^{n} [a_{ij(k_i-1)}, b_{ijk_i}]$$
each having diameter $< d$. Letting S_j be the set of all sub-boxes of B_j, we also have
$$\sum_{b\in S_j} |b| = |B_j|$$
For each j, define
$$S_{jE} = \{b \in S_j | b \cap E \neq \emptyset\}$$
and define S_{jF} similarly. Then clearly
$$\bigcup_{j\in J} S_{jE}$$
covers E, and similar for F. **But S_{jE} and S_{jF} are disjoint** because each box has diameter $< d = \text{dist}(E, F)$. This means that
$$\sum_{b\in S_{jE}} |b| + \sum_{b\in S_{jF}} |b| = \sum_{b\in S_{jE}\cup s_{jF}} |b|$$

Noting that $S_{jE} \cup S_{jF} \subseteq S_j$, we finally obtain

$$\begin{aligned}
m^*(E \cup F) + \varepsilon &> \sum_{j \in J} |B_j| \\
&= \sum_{j \in J} \sum_{b \in S_j} |b| \\
&\geq \sum_{j \in J} \sum_{b \in S_{jE} \cup S_{jF}} |b| \\
&= \sum_{j \in J} \left(\sum_{b \in S_{jE}} |b| + \sum_{b \in S_{jF}} |b| \right) \\
&= \sum_{j \in J} \sum_{b \in S_{jE}} |b| + \sum_{j \in J} \sum_{b \in S_{jF}} |b| \\
&\geq m^*(E) + m^*(F)
\end{aligned}$$

Since ε was arbitrary,

$$m^*(E \cup F) \geq m^*(E) + m^*(F)$$

By finite additivity,

$$m^*(E \cup F) \leq m^*(E) + m^*(F)$$

Cominining,

$$m^*(E \cup F) = m^*(E) + m^*(F)$$

QED.

Lemma 1.

By monotonicity,

$$m^*(A) \leq \inf \{m^*(U) \mid U \supseteq A, U \text{ open}\}$$

Furthermore, letting $\varepsilon > 0$ and $(B_j)_{j \in J}$ be a countable open box cover of A with

$$\sum_{j \in J} |B_j| < m^*(A) + \varepsilon$$

and letting $U \bigcup_{j \in j} B_j$, we have

$$U \subseteq \bigcup_{j \in j} B_j$$

$$\begin{aligned} m^*(U) &\leq \sum_{j \in j} |B_j| \\ &< m^*(A) + \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this gives

$$\inf \{m^*(U) \mid U \supseteq A, U \text{ open}\} \leq m^*(A)$$

$$m^*(A) = \inf \{m^*(U) \mid U \supseteq A, U \text{ open}\}$$

Lemma 2.

Define

$$E = \bigcup_{i=1}^{\infty} E_i$$

We are to show that E is measurable.

Let $\varepsilon > 0$ and pick a sequence of positive numbers $(\varepsilon_i)_{i\in\mathbb{N}}$ such that

$$\sum_{i=1}^{\infty} \varepsilon_i = \varepsilon$$

For each i, pick a $U_i \supseteq E_i$ such that

$$m^*(U_i \setminus E_i) < \varepsilon_i$$

Define

$$U = \bigcup_{i=1}^{\infty} U_i$$

To prove the lemma, we will show that

$$m^*\left(U \setminus E\right) < \varepsilon$$

First, we have

$$\begin{aligned} U \setminus E &= \bigcup_{i\in\mathbb{N}} U_i \setminus \bigcup_{i\in\mathbb{N}} E_i \\ &= \bigcup_{i\in\mathbb{N}} \left(U_i \setminus \bigcup_{i\in\mathbb{N}} E_i \right) \\ &\subseteq \bigcup_{i\in\mathbb{N}} \left(U_i \setminus E_i\right) \end{aligned}$$

By monotonicity and countable subadditivity:

$$\begin{aligned} m^*\left(U \setminus E\right) &\leq m^* \left(\bigcup_{i\in\mathbb{N}} \left(U_i \setminus E_i\right) \right) \\ &\leq \sum_{i\in\mathbb{N}} m^*\left(U_i \setminus E_i\right) \\ &< \sum_{i\in\mathbb{N}} \varepsilon_i \\ &= \varepsilon \end{aligned}$$

QED.

Lemma 3.

Since $\mathbb{R}^n$ is covered by countably many unit cubes, A is a countable union of compact sets:

$$A = \bigcup_{(m_1,\dots m_n)\in\mathbb{Z}^n} \left(A \cap \prod_{i=1}^{n} [m_i, m_i+1] \right)$$

So we assume WLOG that $A \subseteq [0,1]^n$.
Now we make two claims:

1. An open set is a union of countably many closed boxes with disjoint interiors.
2. Outer measure is countably super-additive on closed boxes with disjoint interiors.

First, we show that these claims imply Lemma 3.
Let U be the open set

$$U = (-0.1, 1.1)^n \setminus A$$

and let $(B_i)_{i\in\mathbb{N}}$ be a sequence of closed boxes with disjoint interiors such that

$$\bigcup_{i=1}^{\infty} B_i = U$$

Such a sequence exists by Claim 1.
Observation 1. *Let $A \subseteq U' \subseteq A \cup U$. Let*

$$S = \{ i \in \mathbb{N} \mid U' \cap B_i \neq \emptyset \}$$

Then

$$U' \setminus A \subseteq U$$

$$U' \setminus A \subseteq \bigcup_{i\in S} B_i$$

$$m^*(U' \setminus A) \leq \sum_{i\in S} m^*(B_i)$$

Observation 2. *By Claim 2, $\sum_{i=1}^{\infty} m^*(B_i)$ is finite:*

$$\infty > m^*(U) \geq m^*\left(\bigcup_{i=1}^{\infty} B_i \right) \geq \sum_{i=1}^{\infty} m^*(B_i)$$

Now note that, by compactness and Lebesgue number tactics, for each i there is a radius $r_i > 0$ such that

$$\forall x \in B_i : \quad B_r(x) \subseteq U$$

$$\forall a \in A : \quad B_r(a) \cap B_i = \emptyset$$

($B_r(x)$ is the open ball with radius r and center x.)
Fix some such r_i for each i.
Furthermore, there's a radius $r > 0$ such that

$$\forall a \in A : \quad B_r(a) \subseteq U$$

For each $m \in \mathbb{N}$, define

$$R_m = \min \{ r, r_1, r_2, \dots r_{m-1} \}$$

Then

$$\forall a \in A: \quad B_{R_m}(a) \subseteq U$$
$$\forall a \in A\, \forall i < m: \quad B_{R_m}(a) \cap B_i = \emptyset$$

Now let

$$U'_m = \bigcup_{a \in A} B_{R_m}(a)$$

U'_m satisfies the hypothesis of **Observation 1**, and, letting S be as in **Observation 1**, we find that

$$\forall i \in S : i \geq m$$

This gives

$$m^*(U'_m \setminus A) \leq \sum_{i \in S} m^*(B_i) \leq \sum_{i=m}^{\infty} m^*(B_i)$$

Finally, let $\varepsilon > 0$. By **Observation 2**, there exists $m \in \mathbb{N}$ such that

$$\sum_{i=m}^{\infty} m^*(B_i) < \varepsilon$$

which means that U'_m is an open set containing A such that

$$m^*(U'_m \setminus A) < \varepsilon$$

Hence A is measurable.

We now prove Claims 1 and 2 to complete the proof.

Claim 1. *An open set is a union of countably many closed boxes with disjoint interiors.*

For any $s > 0$ there is a set

$$T_s = \{s\langle m_1, \ldots m_n\rangle + [0, s]^n \mid m_i \in \mathbb{Z}\}$$

which can be said to tile $\mathbb{R}^n$ using translates of $[0, s]^n$ (with disjoint interiors). One might also say that its elements are cubes aligned with grid lines.

This set covers $\mathbb{R}^n$ This set is countable, so any subset of it is countable.

Let U be an open set.

We will fill U with closed cubes having power-of-two sidelengths and aligned with gridlines.

For each $k \in \mathbb{N}_0$, define

$$S_k = \left\{ b \in T_{2^{-k}} \;\middle|\; b \subseteq U, b \not\subseteq \bigcup_{j<k} S_j \right\}$$

For example,

$$S_0 = \{b \in T_1 \mid b \subseteq U\}$$
$$S_1 = \left\{ b \in T_{\frac{1}{2}} \;\middle|\; b \subseteq U, b \not\subseteq S_0 \right\}$$
$$S_2 = \left\{ b \in T_{\frac{1}{4}} \;\middle|\; b \subseteq U, b \not\subseteq S_0 \cup S_1 \right\}$$
$$\ldots$$

Observation 3.

For any integers $i, j \in \mathbb{N}_0$ with $i \leq j$ and any subset $S \subseteq T_{2^{-i}}$, every box b in

$$\left\{ b \in T_{2^{-j}} \;\middle|\; b \not\subseteq \bigcup S \right\}$$

has interior b° disjoint from every b'° for $b' \in S$. For example, the elements of S have disjoint interiors, and the elements of

$$S_\sigma := \bigcup_{k=0}^{\infty} S_k$$

have disjoint interiors.

Note also that S_σ is countable since each S_k is countable.

Clearly

$$\bigcup S_\sigma \subseteq U$$

Furthermore, we will see that

$$U \subseteq \bigcup S_\sigma$$

Indeed, since $\bigcup_{k=0}^{\infty} T_{2^{-k}}$ has boxes with arbitrarily small diameter, it contains a box b with

$$b \subseteq U$$

$$u \in b$$

Taking b such a box with largest possible side length 2^{-k}, we notice that

$$\forall b' \in \bigcup_{j<k} T_{2^{-j}} : \quad u \notin b'$$

Hence

$$b \not\subseteq \bigcup_{j<k} S_j$$

Therefore

$$b \in S_k$$

$$u \in b \subseteq \bigcup S_\sigma$$

$$u \in \bigcup S_\sigma$$

Since u was arbitrary, we now obtain

$$U \subseteq \bigcup S_\sigma$$

$$U = \bigcup S_\sigma$$

Thus, we have finally written U as a countable union of closed boxes with disjoint interiors.

Claim 2. *Outer measure is countably super-additive on closed boxes with disjoint interiors.*

Subclaim. *Let $(B_j)_{j \in J}$ be a finite sequence of closed boxes with disjoint interiors, and B' a closed box. Then $m^*(B) \geq \sum_{j \in J} m^*(B' \cap B_j)$.*

Let $(B_j)_{j \in J}$ be a finite sequence of closed boxes with disjoint interiors, and B' a closed box.

Let E_i be the set of interval endpoints on the i^{th} dimension:

$$E_i = \{a'_i, b'_i\} \cup \bigcup_{j \in J} \{a_{ij}, b_{ij}\}$$

Let

$$B_j = \prod_{i=1}^{n} [a_{ij}, b_{ij}]$$

$$B = \prod_{i=1}^{n} [a'_i, b'_i]$$

For each i, let

$$a_i = \min(E_i)$$

$$b_i = \max(E_i)$$

$$P_i = E_i \cap [a_i, b_i]$$

We regard P_i as a partition of the interval $[a_i, b_i]$, writing

$$P_i = \{a_i = x_{i0} < \cdots < x_{it_i} = b_i\}$$

and we use these partitions to split B naturally into sub-boxes:

$$S = \left\{ \prod_{i=1}^{n} [x_{ik_i-1}, x_{ik_i}] \;\middle|\; 1 \leq k_i \leq t_i \right\}$$

Clearly

$$|B| = \sum_{b \in S} |b|$$

For each $i \in \{1, \dots n\}$ and $j \in J$ there is the sub-partition

$$P_{ij} = P_i \cap [a_{ij}, b_{ij}]$$

which partitions $[a_{ij}, b_{ij}]$.
We let

$$t_{ij} = \max(P_{ij})$$

$$r_{ij} = \min(P_{ij})$$

Now, for each $j \in J$, we take the natural set of sub-boxes

$$\begin{aligned} S_j &= \left\{ \prod_{i=1}^{n} [x_{ik_i-1}, x_{ik_i}] \;\middle|\; r_{ij} \leq k_i \leq t_{ij} \right\} \\ &= \{b \in S \mid b \subseteq B_j\} \\ &= \{b \in S \mid b^\circ \cap B_j \neq \emptyset\} \end{aligned}$$

Clearly

$$|B_j| = \sum_{b \in S_j} |b|$$

Repeat the above constructions and observations for B' by defining P'_i, etc.
Important bit: for any $j \in J$, $S' \cap S_j$ covers $B \cap B_j$. Noting that the S_j are disjoint from

each other and from S', we now see

$$\begin{aligned}
m^*(B') &= |B'| \\
&= \sum_{b\in S'} |b| \\
&= \sum_{b\in S'\setminus\bigcup_{j\in J} S_j} |b| + \sum_{b\in\bigcup_{j\in J} S_j} |b| \\
&\geq \sum_{b\in\bigcup_{j\in J} S_j} |b| \\
&= \sum_{j\in J}\sum_{b\in S_j} |b| \\
&\geq \sum_{j\in J} m^*(B'\cap B_j)
\end{aligned}$$

(The last "$\geq$" is due to **Important bit**.)

This proves the subclaim.

Now let $(B_j)_{j\in J}$ be a finite sequence of boxes, as before, and define

$$C = \bigcup_{j\in J} B_j$$

Pick a $\varepsilon > 0$. Pick a countable cover $(B_k)_{k\in K}$ such that B_k are closed boxes and

$$\sum_{k\in K} |B_k| < m^*(C) + \varepsilon$$

For any $k \in K$, **Subclaim** tells us that

$$m^*(B_k) \geq \sum_{j\in J} m^*(B_k \cap B_j)$$

Noting that each $B_k \cap B_j$ is itself a closed box, we construct a countable closed box cover

$$(B_k \cap B_j)_{k\in K}$$

for each $j \in J$. This gives

$$m^*(B_j) \leq \sum_{k\in K} m^* p B_k \cap B_j$$

Summing over J, we have

$$\begin{aligned}
\sum_{j\in J} m^*(B_j) &\leq \sum_{j\in J}\sum_{k\in K} m^* p B_k \cap B_j \\
&= \sum_{k\in K}\sum_{j\in J} m^* p B_k \cap B_j \\
&\leq \sum_{k\in K} m^*(B_k) \\
&< m^*(C) + \varepsilon
\end{aligned}$$

Since this holds for any $\varepsilon > 0$, we find

$$m^*(C) \geq \sum_{j \in J} m^*(B_j)$$

Indeed, for any $E \supseteq C = \bigcup_{j \in J} B_j$,

$$m^*(E) \geq \sum_{j \in J} m^*(B_j)$$

So, letting $(B_l)_{l \in L}$ be a countable sequence of closed boxes and A their union, we see that for any finite $J \subseteq L$,

$$A \supseteq \bigcup_{j \in J}$$

$$m^*(A) \geq \sum_{j \in J} m^*(B_j)$$

Therefore

$$m^*(A) \geq \sum_{l \in L} m^*(B_j)$$

This finally proves our claim.

QED.

Lemma 4.

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers with infimum 0.
For each n, let U_n be an open set containing E such that

$$m^*\left(U_n \setminus E\right) < a_n$$

Then each U_n^c is closed and

$$m^*\left(\bigcap_{n=1}^{\infty} U_n \setminus E\right) = 0$$

Note now that

$$E^c = \left(\bigcup_{n=1}^{\infty} U_n^c\right) \cup \left(\bigcap_{n=1}^{\infty} U_n \setminus E\right)$$

Since E^c is a countable union of measurable sets, it is measurable.