

25.

(a)

A measurable function f is one whose undergraph $\mathcal{U}f$ is measurable. The completed undergraph $\hat{\mathcal{U}}f$ is then also measurable, so the graph $\hat{\mathcal{U}}f \setminus \mathcal{U}f$ is measurable. The vertical slices of the graph are singleton sets, hence they all have measure 0, so by the zero slice theorem, the graph has measure 0.

(b)

I don't think so. Every indicator function has a zero-measure graph, but I suspect indicator functions for non-measurable sets have non-measurable undergraph. In particular, a proof that Tao's non-measurable set X is non-measurable might work just as well for the set $X \times [0, 1)$.

(c)

(d)

(e)

For any $E \subseteq \mathbb{R}^n$, we define inner measure as:

$$m_*(E) = \sup \{m(A) \mid A \subseteq E \text{ measurable}\}$$

Let G be the graph of a function, and $E \subseteq G$ a measurable subset. Then every vertical slice of E has one or zero elements, and hence is a null set. By the zero slice theorem, every measurable subset of G is a null set, giving

$$m_*(G) = 0$$

(f)

Let $f : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be a function whose graph G is not measurable and therefore has positive outer measure. For each n , $[n, n + 1)$ is measurable, so

$$m^*(G) = \sum_{n \in \mathbb{Z}} m^*(G \cap [n, n + 1))$$

Hence $G \cap [k, k + 1)$ has positive outer measure for some k . Fix k and define

$$f'(x) = f(k + \text{frac}(x))$$

where $\text{frac}(x)$ is the fractional part of x . This function is periodic.

By translation invariance its graph G' has infinite outer measure:

$$m^*(G') = \sum_{n \in \mathbb{Z}} m^*(G \cap [k, k + 1)) = \infty$$

Now, note that for any distinct $r, s \in \mathbb{R}$, the sets

$$\begin{bmatrix} 0 \\ r \end{bmatrix} + G'$$

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are disjoint.

Since there are uncountably many real numbers, this yields uncountably many disjoint subsets of the plane, each with infinite outer measure.

(g)

28.

(a)

(b)

(c)

(d)

(e)