8.2.7.

For each $(a, q) \in \mathbb{N}$, define

$$I_{aq} = \left[\frac{a}{q} - \frac{c}{q^p}\,, \quad \frac{a}{q} + \frac{c}{q^p}\right] \cap [0, 1]$$

Note that $m(I_{aq}) \leq \frac{2c}{q^p}$.
We are asked to show that the following is a null set:

$$\left\{x \in \mathbb{R} \;\middle|\; x \in I_{aq} \text{ for infinitely many } (a, q) \in \mathbb{N}^2\right\}$$

Pick $A \in \mathbb{N}$ such that $A > c + 1$.
For any $(a, q) \in \mathbb{N}^2$,

$$a \geq qA \implies I_{aq} = \emptyset$$

Hence

$$\begin{aligned}
\sum_{(a,q)\in\mathbb{N}^2} m(I_{aq}) &= \sum_{q=1}^{\infty}\sum_{a=1}^{qA} m(I_{aq}) \\
&\leq \sum_{q=1}^{\infty}\sum_{a=1}^{qA} \frac{2c}{q^p} \\
&= \sum_{q=1}^{\infty} qA\frac{2c}{q^p} \\
&= 2Ac\sum_{q=1}^{\infty} \frac{1}{q^{p-1}} \\
&\leq 2Ac\sum_{q=1}^{\infty} \frac{1}{q^2} \\
&= 2Ac\sum_{k=0}^{\infty}\sum_{q=2^k}^{2^{k+1}-1} \frac{1}{q^2} \\
&\leq 2Ac\sum_{k=0}^{\infty}\sum_{q=2^k}^{2^{k+1}-1} \frac{1}{(2^k)^2} \\
&= 2Ac\sum_{k=0}^{\infty} \frac{2^k}{(2^k)^2} \\
&= 2Ac\sum_{k=0}^{\infty} \frac{1}{2^k} \\
&= 4Ac \\
&< \infty
\end{aligned}$$

So, by the Borel-Tonelli theorem,
the given set

$$\left\{x \in \mathbb{R} \;\middle|\; x \in I_{aq} \text{ for infinitely many } (a, q) \in \mathbb{N}^2\right\}$$

has measure zero.

8.2.9.

Let

$$E = \{x \in \mathbb{R} \mid f_n(x) \not\to 0 \text{ as } n \to \infty\}$$

We will show that E has measure 0, thus proving the desired result.
Importantly,

$$\forall x \in E \;\; \exists \varepsilon > 0 \;\; \forall N \in \mathbb{N} \;\; \exists n > N : \quad f_n(x) \geq \varepsilon$$

This yields

$$E = \bigcup_{k=1}^{\infty} \left\{x \in E \;\middle|\; x \in f_n^{-1}\left(\left(\tfrac{1}{k}, \infty\right)\right) \text{ for infinitely many } n\right\}$$

$$m^*(E) \leq \sum_{k=1}^{\infty} m^*\left(\left\{x \in E \;\middle|\; x \in f_n^{-1}\left(\left(\tfrac{1}{k}, \infty\right)\right) \text{ for infinitely many } n\right\}\right)$$

Claim. Let $k \in \mathbb{N}$. Then

$$\left\{x \in E \;\middle|\; x \in f_n^{-1}\left(\left(\tfrac{1}{k}, \infty\right)\right) \text{ for infinitely many } n\right\}$$

is a null set by Borel-Tonelli.

Proof.

Pick $n \in \mathbb{N}$.
Note that $f_n^{-1}\left(\left(\frac{1}{k}, \infty\right)\right)$ is measurable because f_n is measurable and $\left(\frac{1}{k}, \infty\right)$ is open.
Hence the function $\frac{1}{k} \cdot \chi_{f_n^{-1}\left(\left(\frac{1}{k}, \infty\right)\right)}$ is measurable. (In fact, it is simple.)
It minorizes f, and therefore

$$\begin{aligned} m\left(f_n^{-1}\left(\left(\tfrac{1}{k}, \infty\right)\right)\right) \cdot \tfrac{1}{k} &= \int_{\mathbb{R}} \tfrac{1}{k} \cdot \chi_{f_n^{-1}\left(\left(\frac{1}{k}, \infty\right)\right)} \\ &\leq \int_{\mathbb{R}} f_n \\ &\leq \frac{1}{4^n} \end{aligned}$$

$$m\left(f_n^{-1}\left(\left(\tfrac{1}{k}, \infty\right)\right)\right) \leq \frac{k}{4^n}$$

This holds for all $n \in \mathbb{N}$, hence

$$\begin{aligned} \sum_{n=1}^{\infty} m\left(f_n^{-1}\left(\left(\tfrac{1}{k}, \infty\right)\right)\right) &\leq \sum_{n=1}^{\infty} \frac{k}{4^n} \\ &= \frac{k}{3} \\ &< \infty \end{aligned}$$

Borel-Tonelli now proves the claim.

We now see that

$$m^*(E) \leq \sum_{k=1}^{\infty} 0$$

$$m^*(E) = 0$$

QED.

8.2.10.

We state and prove what I believe to be the general case.

Theorem (Egoroff).

Hypotheses:

- (X, S, μ) is a measure space.
- The codomain of μ is $[0, \infty)$. In particular, $\mu(X) < \infty$.
- (M, d) is a metric space.
- $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions $X \to M$ which converge pointwise to a function f, and are measurable (i.e. $f_n^{-1}(V) \in S$ for any open set $V \subseteq M$ and $n \in \mathbb{N}$).

Conclusion: for every $\varepsilon > 0$, there is a set $E \in S$ such that

- $m(E) \leq \varepsilon$
- The sequence f_n converges uniformly on E^c.

Note.

In the given problem,

- $X = [0, 1]$.
- S is the set of Lebesgue-measurable subsets of $[0, 1]$.
- $\mu = m|_S$.
- $M = [0, \infty)$.
- d is the standard metric inherited from $\mathbb{R}$.
- f is the constant 0 function.

Proof.

Let $\varepsilon > 0$ and $\sum_{k=1}^{\infty} \varepsilon_k = \varepsilon$. For each $\gamma > 0, N \in \mathbb{N}$, define

$$F_{\gamma N} = \{x \in X \mid n > N \implies d(f_n(x), f(x)) < \gamma\}$$

$F_{\gamma N}$ is measurable (i.e. a member of S) since

$$F_{\gamma N} = \bigcap_{n>N} f_n^{-1}(B_\gamma(f(x)))$$

Claim. Let $\gamma > 0$. Then

$$\lim_{N\to\infty} \mu(F_{\gamma N}) = \mu(X)$$

Proof.

If $N \leq N'$ then $F_{\gamma N} \subseteq F_{\gamma N'}$. In other words, the sequence $(F_{\gamma N})_{N\in\mathbb{N}}$ is ascending.
Since $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f throughout X, we have

$$\forall x \in X \quad \exists N \in \mathbb{N} : \quad x \in F_{\gamma N}$$

$$\bigcup_{N=1}^{\infty} F_{\gamma N} = X$$

$$\lim_{N\to\infty} \mu(F_{\gamma N}) = \mu(X)$$

Now let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers with infimum 0.
For any sequence of natural numbers $(N_k)_{k\in\mathbb{N}}$,
$(f_n)_{n\in\mathbb{N}}$ converges uniformly on the set

$$\bigcap_{k=1}^{\infty} F_{\gamma_k N_k}$$

Since $\mu(X) < \infty$, for each k there is an N_k such that

$$\mu(F_{\gamma_k N_k}) \geq \mu(X) - \varepsilon_k$$

Choose such an N_k for each k and define

$$E = \left(\bigcap_{k \in \mathbb{N}} F_{\gamma_k N_k} \right)^c$$

Then $(f_n)_{n \in \mathbb{N}}$ converges uniformly on E^c and

$$\begin{aligned} \mu(E) &= \mu\left(\left(\bigcap_{k \in \mathbb{N}} F_{\gamma_k N_k} \right)^c \right) \\ &= \mu\left(\bigcup_{k \in \mathbb{N}} F^c_{\gamma_k N_k} \right) \\ &\leq \sum_{k \in \mathbb{N}} \mu(F^c_{\gamma_k N_k}) \\ &\leq \sum_{k \in \mathbb{N}} \varepsilon_k \\ &\leq \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the theorem is proven.