39.

48.

53.

(a)

Iterated integral 1.

Let $y \in \mathbb{R}$. Then if $y \notin (0,1)$,

$$\forall x \in \mathbb{R} : f(x,y) = 0$$

$$\int_{x\in\mathbb{R}} f(x,y)dx = 0$$

Otherwise,

$$\begin{aligned}\int_{x\in\mathbb{R}} f(x,y)dx &= \int_{x\in(0,1)} f(x,y)dx \\ &= \int_{x\in(0,y)} \frac{1}{y^2}dx + \int_{x\in(y,1)} -\frac{1}{x^2}dx \\ &= \int_{x\in(0,y)} \frac{1}{y^2}dx + \int_{x\in(y,1)} -\frac{1}{x^2}dx \\ &= \frac{1}{y} + (1 - \frac{1}{y}) \\ &= 1\end{aligned}$$

Hence

$$\begin{aligned}\int_{y\in\mathbb{R}}\int_{x\in\mathbb{R}} f(x,y)dxdy &= \int_{y\in(0,1)} 1dy \\ &= 1\end{aligned}$$

Iterated integral 2.

Let $x \in \mathbb{R}$. Then if $x \notin (0,1)$,

$$\forall y \in \mathbb{R} : f(y,x) = 0$$

$$\int_{y\in\mathbb{R}} f(y,x)dy = 0$$

Otherwise,

$$\begin{aligned}\int_{y\in\mathbb{R}} f(y,x)dy &= \int_{y\in(0,1)} f(y,x)dy \\ &= \int_{y\in(0,x)} -\frac{1}{x^2}dy + \int_{y\in(x,1)} \frac{1}{y^2}dy \\ &= \int_{y\in(0,x)} -\frac{1}{x^2}dy + \int_{y\in(x,1)} \frac{1}{y^2}dy \\ &= -\frac{1}{x} + (-1 - (-\frac{1}{x})) \\ &= -1\end{aligned}$$

Hence

$$\int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}} f(y,x)dydx = \int_{x\in(0,1)} -1dx$$
$$= -1$$

Double integral.

We show that $\int f^+$ and $\int f^-$ are both ∞, hence $\int f$ does not exist. For any $k \in \mathbb{N}$, we have

$$(0, \frac{1}{2^k}) \times (\frac{1}{2^k}, \frac{1}{2^{k-1}}) \subseteq (f^+)^{-1}([\frac{2^{2k}}{4}, \infty))$$

Hence

$$S_k := (0, \frac{1}{2^k}) \times (\frac{1}{2^k}, \frac{1}{2^{k-1}}) \times [0, \frac{2^{2k}}{4}) \subseteq \mathcal{U}f^+$$

This gives

$$\begin{aligned}\int f^+ &= m(\mathcal{U}f^+)\\ &\geq m\left(\bigsqcup_{k=1}^{\infty} S_k\right)\\ &= \sum_{k=1}^{\infty} m(S_k)\\ &= \sum_{k=1}^{\infty} \tfrac{1}{4}\\ &= \infty\end{aligned}$$

Similarly,

$$(\frac{1}{2^k}, \frac{1}{2^{k-1}}) \times (0, \frac{1}{2^k}) \subseteq (f^-)^{-1}([\frac{2^{2k}}{4}, \infty))$$

Hence

$$S'_k := (\frac{1}{2^k}, \frac{1}{2^{k-1}}) \times (0, \frac{1}{2^k}) \times [0, \frac{2^{2k}}{4}) \subseteq \mathcal{U}f^+$$

This gives

$$\begin{aligned}\int f^- &= m(\mathcal{U}f^+)\\ &\geq m\left(\bigsqcup_{k=1}^{\infty} S'_k\right)\\ &= \sum_{k=1}^{\infty} m(S'_k)\\ &= \sum_{k=1}^{\infty} \tfrac{1}{4}\\ &= \infty\end{aligned}$$

QED.

(b)

Corollary 43 requires the function to be non-negative, and here, f is sometimes negative.

58.

66.