2.

(a) See the proof below of Egoroff's theorem.

(b) Yes. See the proof below of Egoroff's theorem.

(c)

Let

$$f_n(x) = \frac{1}{n}x$$

$$f(x) = 0$$

$(f_n)_{n\in\mathbb{N}}$ converges pointwise to f everywhere.
But, if $E \subseteq \mathbb{R}$ is unbounded,
then the convergence of $(f_n)_{n\in\mathbb{N}}$ on E is not uniform.
Indeed, let $\varepsilon = 3$ and let $N \in \mathbb{N}$ be given.
Then, letting $n = N + 1$,
we use the unboundedness of E to choose an $x \in E$ with $|x| \geq 3n$.
This gives $n > N$ and

$$\frac{|x|}{n} \geq 3$$

$$\left|\frac{x}{n}\right| \geq 3$$

$$\left|\frac{x}{n} - 0\right| \geq 3$$

$$|f_n(x) - f(x)| \geq \varepsilon$$

Recalling that $N \in \mathbb{N}$ was arbitrary,
we've shown that for any unbounded $E \subseteq \mathbb{R}$,

$$\exists \varepsilon > 0 \;\; \forall N \in \mathbb{N} \;\; \exists x \in E \;\; \exists n > N : \quad |f_n(x) - f(x)| \geq \varepsilon$$

i.e. $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly on E.
So any $E \subseteq \mathbb{R}$ on which $(f_n)_{n\in\mathbb{N}}$ converges uniformly must be bounded, hence its complement has infinite measure, so Egoroff's theorem fails.

(d)

Note: we will only use boundedness of K, not compactness.
Let d be a metric on $\mathbb{R}^n$ and let $(D_i)_{i\in\mathbb{N}}$ be a sequence of measurable subsets of $\mathbb{R}^n$ such that

$$\bigsqcup_{i=1}^{\infty} D_i = \mathbb{R}^n$$

$$s := \inf\{m(D_i) \mid i \in \mathbb{N}\} > 0$$

$$a := \sup\{\operatorname{diam}(D_i) \mid i \in \mathbb{N}\} < \infty$$

(Diameters are taken using d.)
For example, we could take d to be the standard metric
and D_i to be translates of $[0,1)^n$;
this would give $s = 1$ and $a = \sqrt{n}$.

Let $\varepsilon > 0$. Let $\varepsilon_i > 0$ with

$$\sum_{i=1}^{\infty} \varepsilon_i = \varepsilon$$

For each i, pick some $S_i \subseteq D_i$ such that

$$m(S_i) \leq \varepsilon_i$$

$$(f_n)_{n\in\mathbb{N}} \text{ converges uniformly on } D_i \setminus S_i$$

This is possible by Egoroff's.
Letting $S = \bigsqcup_{i=1}^{\infty} S_i$, we have

$$D_i \setminus S_i = D_i \setminus S$$

$$m(S) \leq \varepsilon$$

Let $K \subseteq \mathbb{R}^n$ be bounded (using the metric d) and define

$$I = \{i \in \mathbb{N} \mid D_i \cap K \neq \emptyset\}$$

Lemma. I is finite.
Proof.

Since K is bounded, we may pick $r > 0$ such that

$$B_r(0) \supseteq K$$

This gives

$$\forall i \in \mathbb{N} \;\; \exists x \in D_i : \quad d(0,x) < r$$

Noting that

$$\forall i \in \mathbb{N} \;\; \forall x,y \in D_i : \quad d(x,y) < a$$

we apply the triangle inequality to obtain

$$\forall i \in \mathbb{N} \;\; \forall y \in D_i : \quad d(0,y) < r + a$$

$$\forall i \in \mathbb{N} : \quad D_i \subseteq B_{r+a}(0)$$

This gives

$$\begin{aligned}
\infty &\geq m\left(B_{r+a}(0)\right) \\
&\geq m\left(\bigsqcup_{i\in I} D_i\right) \\
&= \bigsqcup_{i\in I} m(D_i) \\
&\geq \sum_{i\in I} s \\
&= |I| s
\end{aligned}$$

$$\infty > |I| s$$

Since $s > 0$ we may divide by s to obtain

$$\infty > |I|$$

I is finite.

Since I is finite
and $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $D_i \setminus S_i$ for each $i \in I$,
we have

$$(f_n)_{n\in\mathbb{N}} \text{ converges uniformly on } \bigsqcup_{i\in I} D_i \setminus S$$

Hence it converges uniformly on the subset $K \setminus S$.
QED.

Now we prove Egoroff's.
This will be similar to last time.

Definition.

Let (X, Σ, μ) be a measure space and (T, τ) a topological space. A function $f : X \to M$ is **measurable** iff

$$\forall U \in \tau : \quad f^{-1}(U) \in S$$

In words, we require that the preimage of every open set be measurable.

Theorem (Egoroff).

Hypotheses:

- (X, Σ, μ) is a measure space.
- The codomain of μ is $[0, \infty)$. In particular, $\mu(X) < \infty$.
- (M, d) is a metric space.
- $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions $X \to M$ which converge pointwise almost everywhere to a function f, and are measurable.

Conclusion: for every $\varepsilon > 0$, there is a set $S \in \Sigma$ such that

- $m(S) \le \varepsilon$
- $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $X \setminus S$.

Note.

In the given exercise,

- X is an arbitrary box in $\mathbb{R}^n$ (a) or an arbitrary finite-measure subset of $\mathbb{R}^n$ (b).
- Σ is the set of Lebesgue-measurable subsets of X.
- $\mu = m|_\Sigma$.
- $M = \mathbb{R}^m$.
- d is the standard (Euclidean) metric on $\mathbb{R}^m$.

Proof.

Let $\varepsilon > 0$ and $\sum_{k=1}^{\infty} \varepsilon_k = \varepsilon$. For each $\gamma > 0, N \in \mathbb{N}$, define

$$F_{\gamma N} = \{x \in X \mid n > N \implies d(f_n(x), f(x)) < \gamma\}$$

$F_{\gamma N}$ is measurable (i.e. a member of Σ) since

$$F_{\gamma N} = \bigcap_{n>N} f_n^{-1}(B_\gamma(f(x)))$$

Claim. Let $\gamma > 0$. Then

$$\lim_{N\to\infty} \mu(F_{\gamma N}) = \mu(X)$$

Proof.

If $N \leq N'$ then $F_{\gamma N} \subseteq F_{\gamma N'}$. In other words, the sequence $(F_{\gamma N})_{N \in \mathbb{N}}$ is ascending.
Since $(f_n)_{n \in \mathbb{N}}$ converges pointwise to f almost everywhere,
there exists a null set $Z \subseteq X$ such that

$$\forall x \in X \setminus Z \;\; \exists N \in \mathbb{N}: \quad x \in F_{\gamma N}$$

$$\bigcup_{N=1}^{\infty} F_{\gamma N} = X \setminus Z$$

$$\lim_{N \to \infty} \mu(F_{\gamma N}) = \mu(X)$$

Now let $(\gamma_k)_{k \in \mathbb{N}}$ be a sequence of positive numbers with infimum 0.
For any sequence of natural numbers $(N_k)_{k \in \mathbb{N}}$,
$(f_n)_{n \in \mathbb{N}}$ converges uniformly on the set

$$\bigcap_{k=1}^{\infty} F_{\gamma_k N_k}$$

Since $\mu(X) < \infty$, for each k there is an N_k such that

$$\mu(F_{\gamma_k N_k}) \geq \mu(X) - \varepsilon_k$$

Choose such an N_k for each k and define

$$E = \left(\bigcap_{k \in \mathbb{N}} F_{\gamma_k N_k} \right)^c$$

Then $(f_n)_{n \in \mathbb{N}}$ converges uniformly on E^c and

$$\begin{aligned} \mu(E) &= \mu\left(\left(\bigcap_{k \in \mathbb{N}} F_{\gamma_k N_k} \right)^c \right) \\ &= \mu\left(\bigcup_{k \in \mathbb{N}} F^c_{\gamma_k N_k} \right) \\ &\leq \sum_{k \in \mathbb{N}} \mu(F^c_{\gamma_k N_k}) \\ &\leq \sum_{k \in \mathbb{N}} \varepsilon_k \\ &\leq \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the theorem is proven.

3.

Note that

$$||T|| = \sup\{|T\mathbf{x}| \mid \mathbf{x} \in \mathbb{R}^n, |\mathbf{x}| = 1\}$$

$|\cdot|_1$:

$||T||_1$ is the largest 1-norm among column vectors of T, i.e.

$$M := \max\left\{\sum_{i=1}^{n} |T_{ij}| \;\middle|\; 1 \leq j \leq n\right\}$$

Proof: **$||T||_1 \leq M$:**

Let $\mathbf{x} \in \mathbb{R}^n$ with $|\mathbf{x}|_1 = 1$.

$$\begin{aligned}
|T\mathbf{x}|_1 &= |\mathbf{y}|_1 \\
&= \sum_{i=1}^{n} |y_i| \\
&= \sum_{i=1}^{n} \left|\sum_{j=1}^{n} T_{ij}x_j\right| \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n} |T_{ij}||x_j| \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} |T_{ij}||x_j| \\
&= \sum_{j=1}^{n} \left(|x_j| \sum_{i=1}^{n} |T_{ij}|\right) \\
&\leq \sum_{j=1}^{n} |x_j| M \\
&= M \sum_{j=1}^{n} |x_j| \\
&= M|\mathbf{x}|_1 \\
&= M \cdot 1 \\
&= M
\end{aligned}$$

Since the unit vector $\mathbf{x}$ was arbitrary, this gives

$$||T||_1 \leq M$$

$||T||_1 \geq M$:

Let j be such that $\sum_{i=1}^{n} |T_{ij}| = M$.
Define $\mathbf{x}$ so that $x_j = 1$ and all other components of $\mathbf{x}$ are 0.
Then $|\mathbf{x}|_1 = 1$ and

$$|T\mathbf{x}|_1 = \begin{bmatrix} T_{1j} \\ \vdots \\ T_{nj} \end{bmatrix}$$

$$\begin{aligned}||T||_1 &\leq |T\mathbf{x}|_1 \\ &= \sum_{i=1}^{n} |T_{ij}| \\ &= M\end{aligned}$$

QED.

$|\cdot|_{\max}$**:**

$||T||_{\max}$ is the largest 1-norm among row-vectors of T, i.e.

$$M' := \max\left\{ \sum_{j=1}^{n} |T_{ij}| \;\middle|\; 1 \leq i \leq n \right\}$$

Proof:

$||T||_{\max} \leq M'$**:**

Let $\mathbf{x} \in \mathbb{R}^n$ with $|\mathbf{x}|_{\max} = 1$.

$$\begin{aligned}|T\mathbf{x}|_{\max} &= |\mathbf{y}|_{\max} \\ &= \max\left\{|y_i| \mid 1 \leq i \leq n\right\} \\ &= \max\left\{ \left|\sum_{j=1}^{n} T_{ij}x_j\right| \;\middle|\; 1 \leq i \leq n \right\} \\ &\leq \max\left\{ \sum_{j=1}^{n} |T_{ij}||x_j| \;\middle|\; 1 \leq i \leq n \right\} \\ &\leq \max\left\{ \sum_{j=1}^{n} |T_{ij}| \;\middle|\; 1 \leq i \leq n \right\} \\ &= M'\end{aligned}$$

Since the unit vector $\mathbf{x}$ was arbitrary, this gives

$$||T||_{\max} \leq M'$$

$||T||_{\max} \geq M'$**:**

Pick i such that $\sum_{j=1}^{n} |T_{ij}| = M'$

and let $x_j = \operatorname{sgn}(T_{ij})$ for each j.

$$
\begin{aligned}
||T||_{\max} &\geq |T\mathbf{x}|_{\max} \\
&= |\mathbf{y}|_{\max} \\
&\geq |y_i| \\
&= \left| \sum_{j=1}^{n} T_{ij} x_j \right| \\
&= \left| \sum_{j=1}^{n} T_{ij} \operatorname{sgn}(T_{ij}) \right| \\
&= \left| \sum_{j=1}^{n} |T_{ij}| \right| \\
&= \sum_{j=1}^{n} |T_{ij}| \\
&= M'
\end{aligned}
$$

QED.

4.