1.

Existence of partial derivatives.

Let $y_0 \in \mathbb{R}$.

Case 1: $y_0 \neq 0$

The function

$$x \mapsto f(x, y_0) = \frac{xy_0}{x^2 + y_0^2}$$

is a quotient of two differentiable functions, so it is differentiable.
And of course, its derivative at $x = x_0 \in \mathbb{R}$ is precisely

$$\frac{\partial f(x_0, y_0)}{\partial x}$$

So this partial derivative exists for all $x_0 \in \mathbb{R}$.

Case 2: $y_0 = 0$

The function

$$x \mapsto f(x, y_0) = f(x, 0) = 0$$

has derivative 0 everywhere, so

$$\frac{\partial f(x_0, y_0)}{\partial x}$$

exists and is 0 for all $x_0 \in \mathbb{R}$.

Similarly,

$$\frac{\partial f(x_0, y_0)}{\partial y}$$

exists for all $(x_0, y_0) \in \mathbb{R}^2$.

Discontinuity at $(0, 0)$.

The sequence

$$p_n = (\frac{1}{n}, \frac{1}{n})$$

approaches $(0, 0)$ but its image

$$f(p_n) = \frac{1}{2}$$

approaches $\frac{1}{2} \neq f(0, 0) = 0$,
so f is not continuous at $(0, 0)$.

2.

We will show that f is uniformly continuous on E.

Let $p, q \in E$.
Write $v = q - p$ and

$$v = (v_1, \dots v_n)$$

As in Pugh's Theorem 8 proof, define

$$p_j = p + \sum_{k=1}^{j} v_k e_k$$

for $j \in \{0, \dots n\}$, and

$$\begin{aligned} \sigma_j : [0,1] &\to E \\ t &\mapsto p_{j-1} + t v_j e_j \end{aligned}$$

for $j \in \{1, \dots n\}$.
Each function

$$f \circ \sigma_j : [0,1] \to \mathbb{R}$$

is continuous on $[0,1]$ and differentiable on $(0,1)$,
so the mean value theorem yields a $t_j \in (0,1)$ such that

$$(f \circ \sigma_j)'(t_j) = f(p_j) - f(p_{j-1})$$

Noting that

$$(f \circ \sigma_j)'(t_j) = \frac{\partial f(p_{j-1} + t v_j e_j)}{\partial x_j} v_j$$

and using the boundedness of the partial derivatives to define

$$M = \sup \left\{ \left| \frac{\partial f(p')}{\partial x_j} \right| \;\middle|\; p' \in E, j \in \{1, \dots n\} \right\}$$

we find

$$\begin{aligned} d_{\mathbb{R}}(f(q), f(p)) &= |f(q) - f(p)| \\ &= |f(p+v) - f(p)| \\ &= \left| \sum_{j=1}^{n} (f(p_j) - f(p_{j-1})) \right| \\ &= \left| \sum_{j=1}^{n} \frac{\partial f(p_{j-1} + t v_j e_j)}{\partial x_j} v_j \right| \\ &\leq \sum_{j=1}^{n} \left| \frac{\partial f(p_{j-1} + t v_j e_j)}{\partial x_j} \right| |v_j| \\ &\leq \sum_{j=1}^{n} M |v|_{\mathbb{R}^n} \\ &= Mn \cdot d_{\mathbb{R}^n}(p, q) \end{aligned}$$

Note that this holds **for any** $p, q \in E$.
Now, letting $\varepsilon > 0$, we define $\delta = \frac{\varepsilon}{Mn}$
(or, if $M = 0$, we instead let $\delta = 1$), giving

$$d_{\mathbb{R}^n}(p, q) < \delta \implies d_{\mathbb{R}}(f(q), f(p)) < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary and δ works for all $p, q \in E$,
this proves that f is uniformly continuous.

3.

We use an arbitrary metric space (M, d) instead of $\mathbb{R}^2$.

Case 1: $E = \emptyset$.

Define

$$\begin{aligned} f : \mathbb{R}^n &\to \mathbb{R} \\ x &\mapsto 1 \end{aligned}$$

Case 2: $E \neq \emptyset$.

Define

$$\begin{aligned} f : \mathbb{R}^n &\to \mathbb{R} \\ x &\mapsto \operatorname{dist}(x, E) \end{aligned}$$

where

$$\operatorname{dist}(x, E) \coloneqq \inf \{d(x, e) \mid e \in E\}$$

In each case, f trivially satisfies the requirements of the problem (and is, in fact, uniformly continuous). In case 2, we could also define

$$\begin{aligned} f : \mathbb{R}^n &\to \mathbb{R} \\ x &\mapsto \begin{cases} 0 & x \in E \\ \exp(-\frac{1}{\operatorname{dist}(x,E)}) & x \notin E \end{cases} \end{aligned}$$

I suspect that this is smooth when $M = \mathbb{R}^n$.

4.