Homework 3

Samuel Speas

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For these exercises we are working with $\mathbb{R}^n$, and sets and subsets are of $\mathbb{R}^n$. We want to prove the statements of each exercises.

Exercise 1. *Graphs of uniformly continuous functions are measurable, with measure zero.*

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a uniformly continuous function. Then on closed box B, the graph $g = \{(x, f(x)) : x \in B\}$ is measurable with measure zero. As an aside, this statement is true case for continuous functions on a closed box as well, since a continuous function on a compact set is uniformly continuous.

We partition the box B into P, a mesh of closed boxes of side lengths ℓ, so that $B = \bigcup_{p \in P} p$, where P is a set of finitely many translates of the box $\prod_1^N [0, \ell]$. The boxes $\{p \times [\inf f(x),\ \sup f(x)] : x \in p\} = P'$ cover g.

For any $a, b \in B$ every $\epsilon > 0$, there exists $\delta > 0$ such that

$$(*) \qquad \operatorname{dist}(a, b) < \delta \implies |f(a) - f(b)| < \epsilon.$$

Fix some ϵ'. We know that Jordan measure and Lebesgue measure are equivalent on elementary sets (alternatively, without using this fact, take a G_δ set converging to P' and so on). We also know that such measure is unique and independent of partition. Let δ satisfy $(*)$ for $\epsilon = \epsilon'/\operatorname{vol}(B)$ and let $\ell < \delta/\sqrt{n}$. Then this mesh gives

$$m(g) < \operatorname{vol}(B) \frac{\epsilon'}{\operatorname{vol}(B)} = \epsilon'.$$

Let $\epsilon' \to 0$, and the result follows for our closed box B. Letting $\epsilon' = \epsilon/2^{j+1}$ and taking countable union of closed boxes B, or taking union of nested graph sets we see that the result holds for domain R^n.

Exercise 2. *16. If E is measurable set, there exist a G_δ set G, and F_σ set F such that $F \subset E \subset G$, and $m(G, F) = 0$. Conversely, if there is such an $F \subset E \subset G$ then E is measurable.*

21. If $A \subset \mathbb{R}^n$ and $B \subset R^k$ are measurable then $A \times B$ is measurable and $m(A) = m(B)$.

For 16, The result as proven in the book for $\mathbb{R}^2$ also holds for $\mathbb{R}^n$. The reasoning is the same. Consider a closed cube that contains E, and using the measurability of E and definition of outer measure, we see there exist such F_n, G_n such that $m(G_n/F_n) \to 0$ as $n \to \infty$. Taking countable union of subsets with small ϵ measure difference $m(G_n/F_n)$, taking measure of the countable union of boxes of increasing diameter about a point, the limit of the box measure $\mathbb{R}^n$, we see the result holds for unbounded measurable sets, since the countable union of measurable sets is measurable.

For 21, the book again has done all of the necessary steps if we just replace intervals with boxes. Lemma 22 gives that Lemma 23 holds for boxes, we can just replace intervals with boxes in the proof. Lemma 24 similarly holds by replacing with boxes in R^2 with boxes in R^n, and measure zero intervals with measure zero hyperplanes. The proof for theorem 21 follows in line, all reasoning remaining valid for the products of higher dimensional sets.

Exercise 3. *If $A \subset R^n$, then A and $\overline{A}$ have the same Jordan outer measure.*

$A \subset \overline{A}$ so by monotonicity of Jordan outer measure, $m^{*,(J)}(A) \leq m^{*,(J)}(\overline{A})$. Recall that A must be bounded to have a Jordan measure. For every $\epsilon > 0$, we have for finite N a cover $(B_n)_{n\in[N]}$ of A, such that

$$\sum_{n=1}^{N} \operatorname{vol}(B_n) \leq m^{*,(J)}(A) + \epsilon.$$

Since

$$\overline{A} \subset \bigcup_{n\in[N]} (\overline{B_n}), \quad \text{and} \quad \operatorname{vol}(B_n) = \operatorname{vol}(\overline{B_n}),$$

we see that

$$m^{*,(J)}(\overline{A}) \leq \sum_{n\in[n]} \operatorname{vol}(B_n) \leq m^{*,(J)}(A) + \epsilon.$$

Letting $\epsilon \to 0$, we have

$$m^{*,(J)}(\overline{A}) \leq m^{*,(J)}(A),$$

and therefore $m^{*,(J)}(A) = m^{*,(J)}(\overline{A})$. $\overline{A}$ is compact so $m^{*,(J)}(\overline{A}) = m^*(\overline{A}) = m(\overline{A})$.