01/18/2022

MATH 105 Discussion (01/18)

1. Consider the open intervals $(1-\frac{\varepsilon}{6}, 1+\frac{\varepsilon}{6})$, $(2-\frac{\varepsilon}{6}, 2+\frac{\varepsilon}{6})$, $(3-\frac{\varepsilon}{6}, 3+\frac{\varepsilon}{6})$
$\{1,2,3\}$ is a subset of the union of these ~~sets~~ intervals.
Hence, $m^*(\{1,2,3\}) \leq 3\cdot 2\cdot \frac{\varepsilon}{6} = \varepsilon$
Since ε is arbitrarily chosen, $\boxed{m^*(\{1,2,3\})=0}$ m^* is lower bounded by 0.

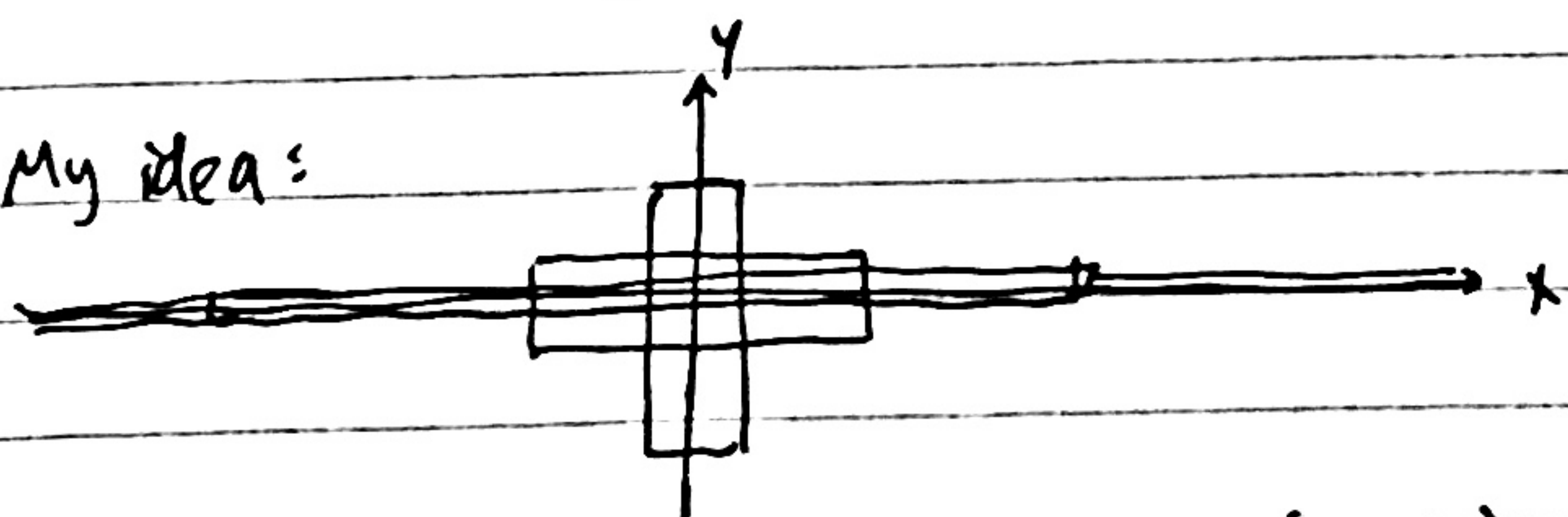
2. For each $z \in \mathbb{Z}$, consider the interval $(z-\frac{\varepsilon}{4\cdot 2^{|z|}}, z+\frac{\varepsilon}{4\cdot 2^{|z|}})$.
Then, $m^*(\mathbb{Z}) < 2\cdot 2\left(\frac{\varepsilon}{4\cdot 2}+\frac{\varepsilon}{4\cdot 2^2}+\cdots\right)$
$= \varepsilon$
Since ε is arbitrarily chosen, $\boxed{m^*(\mathbb{Z})=0}$

For $\mathbb{Q}$, we can enumerate it as it is countably infinite. Since $\mathbb{N}$ has outer measure 0, the same interval used to cover an element of $\mathbb{N}$ can be translated to cover the corresponding element of $\mathbb{Q}$ $\Rightarrow$ $\boxed{m^*(\mathbb{Q})=0.}$

3. The countable union of sets of outer measure 0 has outer measure 0.
Let the sets with outer measure 0 be $A_1, A_2, \ldots$
Then since $m^*(A_i)=0$, in particular, $\forall \varepsilon > 0$, it is not a lower bound
$\Rightarrow \exists$ a covering of A_i with sum of length $< \frac{\varepsilon}{2^i}$.

Thus, the union of these coverings will cover $A_1 \cup A_2 \cup \cdots$ with length $< \frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\cdots = \varepsilon$
Since ε arbitrarily chosen, $m^*(A_1 \cup A_2 \cup \cdots) = 0$.

4. My idea:



Intuitively, decrease the height and make width longer.
open rectangles with area $\frac{\varepsilon}{2^n}$

Consider the rectangles $\cancel{(-\frac{1}{2},\frac{1}{2})\times(-\frac{\varepsilon}{4},\frac{\varepsilon}{4})}$, $\cancel{(-\frac{1}{4},\frac{1}{4})\times(\varepsilon, \varepsilon}$
$(-1,1)\times(-\frac{\varepsilon}{8},\frac{\varepsilon}{8})$, $(-2,2)\times(-\frac{\varepsilon}{32},\frac{\varepsilon}{32})$, $(-4,4)\times(-\frac{\varepsilon}{128},\frac{\varepsilon}{128})$, ...
i.e. consider $B_i = (-2^i, 2^i) \times \left(-\frac{\varepsilon}{2^{3+2i}}, \frac{\varepsilon}{2^{3+2i}}\right)$
$0 \leq i$
Then, the sum of areas of these $B_i = \frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\cdots = \varepsilon$.
Hence $m^*(\mathbb{R}) \leq \varepsilon$ $\forall \varepsilon \geq 0$ $\therefore$ $\boxed{m^*(\mathbb{R})=0}$